Banach J. Math. Anal. 4 (2010), no. 1, 53–86

Banach Journal of Mathematical Analysis
ISSN: 1735-8787 (electronic)
www.emis.de/journals/BJMA/

EXISTENCE, UNIQUENESS AND STATISTICAL THEORY OF TURBULENT SOLUTIONS OF THE STOCHASTIC NAVIER–STOKES EQUATION, IN THREE DIMENSIONS, AN OVERVIEW

BJÖRN BIRNIR[1]

This paper is dedicated to Lars-Erik Persson on the occasion of his 60th birthday

Communicated by P. Wall

Abstract. We discuss the proofs of the existence and uniqueness of solutions of the Navier–Stokes equation driven with additive noise in three dimensions, in the presence of a strong uni-directional mean flow with some rotation. We also discuss how the existence of a unique invariant measure is established and the properties of this measure are described. The invariant measure is used to prove Kolmogorov's scaling in 3-dimensional turbulence including the celebrated $-5/3$ power law for the decay of the power spectrum of a turbulent 3-dimensional flow. Then we briefly describe the mathematical proof of Kolmogorov's statistical theory of turbulence.

1. Introduction

It is well known among fluid dynamicists and natural scientists that the vast majority of fluid flows in nature are turbulent, see for example Monin and Yaglom [20, 21]. Even small streams can have Reynold numbers in the thousands and a typical river has Reynolds number in the range from a hundred thousand to a million, see [10] and [6]. The Reynolds number is the unitless quantity used to

Date: Received: 27 November 2009; Accepted: 31 January 2010.
2000 *Mathematics Subject Classification.* Primary 35R60; Secondary 35R60, 76F02, 76F55.
Key words and phrases. Turbulence, uniqueness, invariant measures, blow-up, stochastic equation.

characterize fluid flow. It is

$$R = \frac{UL}{\nu},$$

where U is a typical velocity, L is a typical length scale in the flow and ν is the viscosity of the fluid. The transition from laminar to turbulent flow starts above $R = 500$ and a typical flow in nature is usually fully turbulent, when $R = 2000$.

It is also clear to the practicioners in the field that there is a big difference between laminar and turbulent flow. Whereas the trajectories of the fluid particles are regular and can be followed in laminar flow, they are very irregular and hard to follow in turbulent flow. Fluid structures are rare or simple in laminar flow whereas they are numerous and exist on all scales in turbulent flow. The list of these differences goes on and it is evident that whereas laminar flow is predictable, turbulent flow is not and this is the reason why engineers and physicists working with turbulent fluids typically use statistical methods to characterize the flow.

It is also abundantly clear how the irregular structure in turbulent flow originates. It is initiated by noise even very small white noise that is always present in fluid flow. This small noise is magnified in turbulent flow by unstable fluid structures but it is quelled in laminar flow. There exists an abundance of fluid experiments were the noise is reduced by modern experimental methods. This reduction delays the onset of turbulence, see for example [9, 8], but it rarely prevents it.

In this paper we will give an overview of the mathematical methods that have recently been used to prove the existence and uniqueness of solutions of the stochastic Navier–Stokes equation and the existence of a unique invariant measure for this equation. We also discuss how the measure can be computed and used to analyze the properties of the noise in turbulent fluid flow. This type of noise then drives the stochastic Navier–Stokes equation and this completes the mathematical proof of Kolmogorov's statistical theory of turbulence.

In the Lagrangian formulation the flow of a small fluid particle with coordinates $X(t)$ is determined by the equation

$$\frac{dX}{dt} = u(X(t), t) \tag{1.1}$$

In turbulent flow the path of the fluid particle is going to be influenced by turbulent noise and the resulting trajectory of the fluid particle is going to resemble a random walk. It is reasonable to assume that the velocity u is in fact a random variable and that it satisfies a stochastic equation that can be written as

$$du = \frac{\partial u}{\partial t} dt + df_t$$

Here $\frac{\partial u}{\partial t}$ is the deterministic acceleration of the fluid and df_t is a random force modeling the influence of the random fluctuations in turbulent flow on the velocity. If we now substitute the right hand side of the deterministic Navier–Stokes in for the time derivative of u in the equation (1.2) we get the stochastically driven Navier–Stokes equation

$$du = (\nu \Delta u - u \cdot \nabla u - \nabla p)dt + df_t \tag{1.2}$$

with the incompressibility condition

$$\nabla \cdot u = 0$$

This is the equation that we will analyze in this paper. Once we have solved it for the stochastic velocity $u(x,t)$, u can be substituted into the equation (1.1) for the random motion of the fluid particle.

Kolmogorov's theory of turbulence published in 1941 [14] set the stage for the resolution of one of the oldest problems in modern mathematics, that of the mathematical formulation of the equations for turbulent flow and their statistical solution. However, to provide a rigorous derivation of Kolomgorov's statistical theory of turbulence has proven to be elusive. This has held back improvements of many application of his theory including application to numerical simulation of turbulent flow. The detailed mathematical theory that we describe in this paper is expected to have major applications to current technology once all its ramifications have been fully developed.

To prove Kolmogorov's theory we must model the noise term, and following [5] we will make the assumption

$$df_t = \sum_{k\neq 0} h_k^{1/2} d\beta_t^k e_k \tag{1.3}$$

in this paper. This assumes that in the statistically stationary state the system is driven by noise (fluctuations) that characterizes a balance between the noise producing (amplifying) nonlinear terms in (1.2). This is a common assumption by investigators in this field, see for example [26, 16, 18]. Here the $e_k s$ are basis vectors that can be taken to be Fourier coefficients, they each come with an independent Brownian motion β_t^k and the $h_k^{1/2}$ are decay vectors that depend on the characteristics of the flow. In particular, this assumes that the variance of the noise

$$E(\langle df_t, df_t\rangle)$$

is finite. This form of the noise assumes that the motion of the fluid particles is continuous, an assumption that makes sense on physical grounds. If the fluid particles moved only under the influence of this noise their velocity would execute an infinite-dimensional Brownian motion.

Hopf [12] found an equation determining the invariant measure in turbulence. This measure can be computed, see [7], and analyzed and it provides the missing detail in Kolmogorov's theory namely the detailed form of the noise (1.3). We give a simplified version of this argument in section 7.

In spite of the rotating vector field that we assume constitutes the largest structure in the flow, the problem solved in this paper is very different from that solved by Babin, Mahalov and Nicolaenko in [1] and [2]. In their papers the rotation plays the main role whereas the uni-directional flow, along the axis of the rotation, is the main actor in this paper. It causes oscillations that permit us to prove the global existence and uniqueness. In this paper the rotation is present for a purely technical reason, to control the velocity components orthogonal to the uniform flow. The two problems are similar in that the initial flow is unstable and the turbulent flow becomes three dimensional. However, in [1] and [2] the three dimensional energy cascade is suppressed and instead there is an inverse cascade similar to two-dimensional flow, whereas in our work the full three-dimensional energy cascade is present and plays a major role in the turbulence production and transfer of energy.

2. Noise and the stochastic initial value problem

The Reynolds number

$$R = \frac{UL}{\nu}$$

has to be over 2000 to get fully developed turbulence as we discussed in the introduction. This means that we can assume that the velocity U of the flow is sufficiently large. Indeed we can assume that the velocity of the flow is of the form

$$w = \mathbf{U} + u\,,$$

where $\mathbf{U}$ is a prescribed flow (vector) and establish the existence of the correction u, which constitutes the turbulent part of the velocity. This is a perturbative approach but u is not necessarily small. It can typically be as large, but not larger, than $\mathbf{U}$.

We will denote the mean flow in the fully developed turbulent state by U_1 and assume that uniform flow with rotation is of the form

$$\frac{\partial x}{\partial t} = \mathbf{U} = U_1 j_1 - A\sin(\Omega t + \theta_0) j_2 + A\cos(\Omega t + \theta_0) j_3\,, \tag{2.1}$$

where the rotation can be extended in a periodic fashion from $\mathbb{T}^3$ to $\mathbb{R}^3$.[1] One can also extend a convection cell pattern from four copies of $\mathbb{T}^3$ to $\mathbb{R}^3$ but we will not use that in this paper. This implies that the deterministic particle motion in the rotating uniform flow is simply

$$x(t) = [U_1 j_1 + \frac{A}{\Omega}\cos(\Omega t + \theta_0) j_2 + \frac{A}{\Omega}\sin(\Omega t + \theta_0) j_3]$$

By the same reasoning as above we can choose the coordinates so that the mean flow component $U_1 j_1$ (2.1) is in the x_1 direction and this direction is the axis of the rotation.

First consider the stirred Navier–Stokes equation

$$\begin{aligned} w_t + w\cdot\nabla w &= \nu\Delta w - \nabla\Delta^{-1}\mathrm{trace}(\nabla w)^2 \\ &\quad -A\Omega\cos(\Omega t + \theta_0) j_2 - A\Omega\sin(\Omega t + \theta_0) j_3 \\ w(x,0) &= U_1 j_1 - A\sin(\theta_0) j_2 + A\cos(\theta_0) j_3\,, \end{aligned} \tag{2.2}$$

where we have used incompressibility conditions

$$\nabla\cdot w = 0$$

to eliminate the pressure term. We want to consider turbulent flow driven by a unidirectional mean flow and to do that we consider the flow to be in a box and impose periodic boundary conditions on the box. Since we are mostly interested

[1]For physical applications, see [13], cylindrical coordinates are more appropriate but cumbersome.

in what happens in the direction along the unidirectional flow we take our x_1 axis to be in that direction. The source of the small (white) noise can be thought of as fluctuations in the stirring rate of the uniform flow in equation (2.2).

The corresponding stochastic Navier–Stokes equation can be written as

$$\begin{aligned} du = (\nu\Delta u - U_1\partial_{x_1}u + A\sin(\Omega t+\theta)\partial_{x_2}u - A\cos(\Omega t+\theta)\partial_{x_3}u \\ -u\cdot\nabla u + \nabla\Delta^{-1}[\mathrm{trace}(\nabla u)^2])dt + \sum_{k\neq 0} h_k^{1/2} d\beta_t^k e_k\,, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \frac{\partial u}{\partial t} + U_1\partial_{x_1}u - A\sin(\Omega t+\theta)\partial_{x_2}u + A\cos(\Omega t+\theta)\partial_{x_3}u + u\cdot\nabla u \\ = \nu\Delta u + \nabla\Delta^{-1}[\mathrm{trace}(\nabla u)^2] \end{aligned}$$

is the driven Navier–Stokes equation (2.2) for $u = w - U_1 j_1 + A\sin(\Omega t+\theta)j_2 - A\cos(\Omega t+\theta)j_3$. $U_1 j_1$ is the now the constant mean flow of the (fully developed) turbulent fluid and $\sum_{k\neq 0} h_k^{1/2} d\beta_t^k e_k$ models the noise in fully developed turbulent flow. We will take the initial condition to be zero $u(x,0)=0$ for convenience and assume that the incompressibility condition

$$\nabla\cdot u(x,t) = 0$$

is satisfied. However, the problem is just as easily solved with a nontrivial initial condition, see Theorem 5.2.

The goal is to prove the existence of a unique solution to (2.3) but also to determine the smoothest space, where these solutions can live because this will determine the decay of the coefficients $h_k^{1/2}$ in the turbulent noise in (2.3). The noise that we end up with will model the intrinsic noise in turbulence and the model is confirmed both by numerical simulations [27], [19] and a direct calculation of the intrinsic noise in turbulence, see [7], as discussed in the introduction.

The first question one might ask about the equation (2.3) is how the noise got introduced into the equation, see [3]. To answer that question consider the Navier–Stokes equation

$$w_t + w\cdot\nabla w = \nu\Delta w + \nabla\{\Delta^{-1}[\mathrm{trace}(\nabla w)^2]\}$$

and linearize it about the divergence-free initial flow $\mathbf{U} = U_0 j_1 + U'(x_1, -\frac{x_2}{2}, -\frac{x_3}{2})^T$. Here T denotes transpose and $\mathbf{U}$ is construed to be the periodic extension of the above formula from $\mathbb{T}^3$ to $\mathbb{R}^3$,

$$\begin{aligned} & u_t + U_0\partial_{x_1}u + U'\begin{pmatrix} u_1 \\ -\frac{u_2}{2} \\ -\frac{u_3}{2}\end{pmatrix} + U'\begin{pmatrix} x_1 \\ -\frac{x_2}{2} \\ -\frac{x_3}{2}\end{pmatrix}\cdot\nabla u + U'U_0 j_1 + (U')^2\begin{pmatrix} x_1 \\ \frac{x_2}{4} \\ \frac{x_3}{4}\end{pmatrix} \\ = \; & \nu\Delta u + \nabla\Delta^{-1}(\frac{3}{2}U'^2 + 2U'(\partial_{x_1}u_1 - \partial_{x_2}u_2 - \partial_{x_3}u_3)) \\ & u(x,0) = 0 \end{aligned}$$

We assume that there is small noise

$$df^0 = \sum_{k\neq 0} c_k^{1/2} d\beta_t^k e_k \tag{2.4}$$

present in the fluid. Note that the coefficients $c_k^{1/2} \neq h_k^{1/2}$ are small. Then u satisfies the linear stochastic PDE

$$\begin{aligned} du &= [\nu\Delta u - U_0\partial_{x_1} u - U'\begin{pmatrix} u_1 \\ -\frac{u_2}{2} \\ -\frac{u_3}{2}\end{pmatrix} - U'\begin{pmatrix} x_1 \\ -\frac{x_2}{2} \\ -\frac{x_3}{2}\end{pmatrix}\cdot\nabla u - U'U_0 j_1 \\ &\quad -(U')^2\begin{pmatrix} x_1 \\ \frac{x_2}{4} \\ \frac{x_3}{4}\end{pmatrix}\cdot\nabla u + \nabla\Delta^{-1}(\frac{3}{2}U'^2 + 2U'(\partial_{x_1}u_1 - \partial_{x_2}u_2 - \partial_{x_3}u_3))]dt \\ &\quad + \sum_{k\neq 0} c_k^{1/2} d\beta_t^k e_k\,, \end{aligned}$$

where the term $\sum_{k\neq 0} c_k^{1/2} d\beta_t^k e_k$ represents stochastic forcing by the small ambient noise.

The solution of this linear equation can be found by use of a Fourier series and it is

$$\begin{aligned} u(x,t) &= \sum_{k\neq 0}\int_0^t e^{-(4\nu\pi^2|k|^2 + 2\pi i U_0 k_1)(t-s)} \\ &\quad \times\left(c_k^{1/2}(1)e^{-U'(t-s)}j_1 + c_k^{1/2}(2)e^{\frac{U'}{2}(t-s)}j_2 + c_k^{1/2}(3)e^{\frac{U'}{2}(t-s)}j_3\right)d\beta_t^k e_k \\ &\quad + O(|U'|)\,, \end{aligned}$$

where $c_k^{1/2}(i), i = 1,2,3$ denotes the ith entry of the three vector $c_k^{1/2}$. Now the expectation of $u(x,t)$ vanishes but the variation is

$$\begin{aligned} E(|u|_2^2)(t) &= \sum_{k\neq 0}\int_0^t e^{-8\nu\pi^2|k|^2(t-s)}(c_k(1)e^{-2U'(t-s)} \\ &\quad + c_k(2)e^{U'(t-s)} + c_k(3)e^{U'(t-s)})ds + O(|U'|^2)\,. \end{aligned}$$

This shows that one one hand the small noise will grow exponentially in time, in the $e_k j_1$ direction, if

$$U' < 0$$

and if $|U'| > 8\pi^2\nu|k|^2$ for some $k \in \mathbb{Z}^3 \setminus \{0\}$, but $|U'|$ is small compared to the exponentially growing term. If on the other hand

$$U' > 0$$

the small noise will grow exponentially in the $e_k j_2$ and $e_k j_3$ directions (in function space), again with $|U'|$ small compared to the exponentially growing term.

The exponential growth of the noise will, however, only continue for a limited time. The growth is quickly saturated by the nonlinear terms in the equation and fluid becomes fully turbulent.

The initial value problem (2.3) can also be written as an integral equation

$$u(x,t) = u_o(x,t) - \int_0^t K(t-s) * (u \cdot \nabla u - \nabla \Delta^{-1}[\text{trace}(\nabla u)^2]) ds \,, \qquad (2.5)$$

where K is the (oscillatory heat) kernal

$$K * f = \sum_{k \neq 0} \int_0^t e^{-(4\pi^2|k|^2 + 2\pi i U_1 k_1)(t-s) - 2\pi i A(k_2,k_3)[\sin(\Omega t+\theta) - \sin(\Omega s+\theta)]} \hat{f}(k,s) ds e_k, \qquad (2.6)$$

$$A(k_2,k_3) = A\sqrt{k_2^2 + k_3^2}, \quad \theta = \tan^{-1}(\frac{k_2}{k_3}) - \theta_0$$

and

$$u_o(x,t) = \sum_{k \neq 0} h_k^{1/2} \int_0^t e^{-(4\pi^2|k|^2 + 2\pi i U_1 k_1)(t-s) - 2\pi i A(k_2,k_3)[\sin(\Omega t+\theta) - \sin(\Omega s+\theta)]} d\beta_s^k e_k(x) \qquad (2.7)$$

is a sum of independent oscillatory processes,

$$A_t^k = \int_0^t e^{-(4\pi^2|k|^2 + 2\pi i U_1 k_1)(t-s) - 2\pi i A(k_2,k_3)[\sin(\Omega t+\theta) - \sin(\Omega s+\theta)]} d\beta_s^k \qquad (2.8)$$

with mean zero, see for example [25]. These processes are reminiscent of Ornstein–Uhlenbeck processes and we will call them oscillatory Ornstein–Uhlenbeck-type processes below.

The mean (average) of the solution u_o of the linear equation is zero by the formula (2.7) and this implies that the solution u of (2.3) also has mean (average) zero

$$\bar{u}(t) = \int_{\mathbb{T}^3} u(x,t) dx = 0$$

This also implies that

$$|w|_2^2 = |\mathbf{U}|^2 + |u|_2^2 \qquad (2.9)$$

for $w = \mathbf{U} + u$ and $\mathbf{U} = U_1 j_1 - A\sin(\Omega t + \theta_0) j_2 + A\cos(\Omega t + \theta_0) j_3$ with $|\mathbf{U}| = \sqrt{U_1^2 + A^2}$. We will derive apriori estimates for w in the next section but then apply them to u in subsequent section using (2.9).

3. Some function spaces and Leray theory

In this section we will explain the probabilistic setting and prove some a priori estimates.

We let $(\Omega, \mathcal{F}, \mathbb{P})$, Ω is a set (of events) and $\mathcal{F}$ a σ-algebra on Ω, denote a probability space with $\mathbb{P}$ the probability measure of Brownian motion and $\mathcal{F}_t$ a filtration generated by all the Brownian motions β_t^k on $[t, \infty)$. If $f : \Omega \to H$ is a random variable, mapping Ω into a Hilbert space H, for example $H = L^2(\mathbb{T}^3)$, then $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$ is a Hilbert space with norm:

$$\|f\|^2_{L^2(\Omega,\mathcal{F},\mathbb{P};H)} = E(|f(\omega)|_2^2) = \int_\Omega |f(\omega)|_2^2 \mathbb{P}(d\omega) = \int_H |x|^2 f_\# \mathbb{P}(dx) \,,$$

where E denotes the expectation with respect to $\mathbb{P}$ and $f_{\#}\mathbb{P}$ denotes the pull-back of the measure $\mathbb{P}$ to H. A stochastic process f_t in $\mathcal{L}^2 = L^2([0,T]; L^2(\Omega, \mathcal{F}, \mathbb{P}; H))$ has the norm

$$\|f_t\|^2_{\mathcal{L}^2} = \int_0^T E(|f(t,\omega)|_2^2)dt$$

and f_t has the following properties, see Oksendal [22].

Definition 3.1.

(1) $f(t,\omega) : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ is measurable with respect to $\mathcal{B} \times \mathcal{F}$, where $\mathcal{B}$ is the σ-algebra of the Borel sets on $[0,\infty)$, $\omega \in \Omega$,
(2) $f(t,\omega)$ is adapted to the filtration $\mathcal{F}_t$,
(3)
$$E(\int_0^T f^2(t,\omega)dt) < \infty.$$

We are mostly interested in the Hilbert spaces $H = H^m(\mathbb{T}^3) = W^{(m,2)}$ that are the Sobolev spaces based on L^2 with the Sobolev norm

$$\|u\|_m^2 = |(1-\Delta^2)^{m/2}u|_2^2$$

The corresponding norm on $\mathcal{L}^2_m = L^2([0,T]; L^2(\Omega, \mathcal{F}, \mathbb{P}; H^m(\mathbb{T}^3)))$ is

$$\|u\|_{\mathcal{L}^2_m} = \left[\int_0^T E(\|u\|_m^2)dt\right]^{1/2}$$

We will abuse notation slightly in this section by writing u instead of w. This is done for future reference and an easier comparison with Leray's classical estimates.

Let $\langle\cdot,\cdot\rangle$ denote the inner product on $L^2(\mathbb{T}^3)$. The following a priori estimates provide the foundation of the probabilistic version of Leray's theory.

Lemma 3.2. *The L^2 norms $|u|_2(\omega,t)$ and $|\nabla u|_2(\omega,t)$ satisfy the identity*

$$d|u|_2^2 + 2\nu|\nabla u|_2^2 dt = 2\sum_{k\neq 0}\langle u, h_k^{1/2}e_k\rangle d\beta_t^k + \sum_{k\neq 0} h_k dt \tag{3.1}$$

and the bounds

$$\begin{aligned} |u|_2^2(\omega,t) &\leq |u|_2^2(0)e^{-2\nu\lambda_1 t} + 2\sum_{k\neq 0}\int_0^t e^{-2\nu\lambda_1(t-s)}\langle u, h_k^{1/2}e_k\rangle d\beta_s^k \\ &\quad + \frac{1-e^{-2\nu\lambda_1 t}}{2\nu\lambda_1}\sum_{k\neq 0} h_k \qquad (3.2) \\ \int_0^t |\nabla u|_2^2(\omega,s)ds &\leq \frac{1}{2\nu}(|u|_2^2(0) - |\mathbf{U}|^2) + \frac{1}{\nu}\sum_{k\neq 0}\int_0^t \langle u, h_k^{1/2}e_k\rangle d\beta_s^k + \frac{t}{2\nu}\sum_{k\neq 0} h_k\,, \end{aligned}$$

where λ_1 is the smallest eigenvalue of $-\Delta$ with vanishing boundary conditions on the box $[0,1]^3$ and $h_k = |h_k^{1/2}|^2$. $\mathbf{U}$ is the velocity vector from the previous section.

The expectations of these norms are also bounded

$$E(|u|_2^2)(t) \quad \leq \quad E(|u|_2^2(0))e^{-2\nu\lambda_1 t} + \frac{1-e^{-2\nu\lambda_1 t}}{2\nu\lambda_1}\sum_{k\neq 0} h_k \tag{3.3}$$

$$E(\int_0^t |\nabla u|_2^2(s)ds) \quad \leq \quad \frac{1}{2\nu}[E(|u|_2^2(0)) - |\mathbf{U}|^2] + \frac{t}{2\nu}\sum_{k\neq 0} h_k \tag{3.4}$$

Proof. The identity (3.1) follows from Leray's theory and Ito's Lemma. We apply Ito's Lemma to the L^2 norm of u squared,

$$d\int_{\mathbb{T}^3} |u|^2 dx = 2\int_{\mathbb{T}^3} \frac{\partial u}{\partial t}\cdot u dx dt + 2\sum_{k\neq 0}\int_{\mathbb{T}^3} u\cdot h_k^{1/2} e_k dx d\beta_t^k + \sum_{k\neq 0} h_k \int_{\mathbb{T}^3} dx dt\,,$$

where $k \in \mathbb{Z}^3$ and $h_k^{1/2} \in \mathbb{R}^3$. Now by use of the Navier–Stokes equation (2.2)

$$\begin{aligned} d|u|_2^2 \quad &= \quad 2\int_{\mathbb{T}^3} \nu\Delta u\cdot u + (-u\cdot\nabla u + \nabla\Delta^{-1}(\mathrm{trace}(\nabla u)^2)\cdot u dx dt \\ &\quad\quad +2\sum_{k\neq 0}\int_{\mathbb{T}^3} u\cdot h_k^{1/2} e_k dx d\beta_t^k + \sum_{k\neq 0} h_k dt \\ &= \quad -2\nu|\nabla u|_2^2 dt + 2\sum_{k\neq 0}\int_{\mathbb{T}^3} u\cdot h_k^{1/2} e_k dx d\beta_t^k + \sum_{k\neq 0} h_k dt \end{aligned}$$

since the divergent-free vector u is orthogonal both to $u\cdot\nabla u$ and the gradient $\nabla\Delta^{-1}(\mathrm{trace}(\nabla u)^2)$ by the divergence theorem. Notice that the inner product (average) of u and the stirring force f in equation (2.2) vanishes, $\langle u, f\rangle = \bar{u}\cdot f = 0$, so f can be omitted in the computation. The first term in the last expression is obtained by integration by parts. This is the identity (3.1). The inequality (3.2) is obtained by applying Poincaré's inequality

$$\lambda_1 |u|_2^2 \leq |\nabla u|_2^2\,,$$

where λ_1 is the smallest eigenvalue of $-\Delta$ with vanishing boundary conditions on the cube $[0, 1]^3$. [2] By Poincaré's inequality

$$d|u|_2^2 + 2\nu\lambda_1|u|_2^2 dt \leq d|u|_2^2 + 2\nu|\nabla u|_2^2 dt$$

$$= 2\sum_{k\neq 0}\langle u, h_k^{1/2} e_k\rangle d\beta_t^k + \sum_{k\neq 0} h_k dt$$

Solving the inequality gives (3.2). (3.3) is obtained by integrating (3.1)

$$|u|_2^2(t) + 2\nu\int_0^t |\nabla u|_2^2(s)ds = |u|_2^2(0) + 2\sum_{k\neq 0}\int_0^t \langle u, h_k^{1/2} e_k\rangle d\beta_s^k + t\sum_{k\neq 0} h_k$$

and dropping $|u-\mathbf{U}|_2^2(t) > 0$, by use of (2.9).

[2] We should subtract the mean from u in Poincaré's inequality because of the periodic boundary conditions, but the mean just washes out in the estimates.

Finally we take the expectations of (3.2) and (3.3) to obtain respectively (3.3) and (3.4), using that the function $\langle u, h_k^{1/2} e_k\rangle(\omega, t)$ is adapted to the filtration $\mathcal{F}_t$. $\square$

The following amplification of Leray's a priori estimates will play an important role in the a priori estimates of the solution of the stochastic Navier–Stokes equation below.

Lemma 3.3. *Let $u_{\frac{1}{2B}} = u(x, t + \frac{1}{2B})$ denote the translation of u in time by the number $\frac{1}{2B}$. Then the L^2 norms of the differences $|u - u_{\frac{1}{2B}}|_2(\omega, t)$ and $|\nabla u - \nabla u_{\frac{1}{2B}}|_2(\omega, t)$ satisfy the identity*

$$d|u - u_{\frac{1}{2B}}|_2^2 + 2\nu|\nabla u - \nabla u_{\frac{1}{2B}}|_2^2 dt = 2\sum_{k\neq 0}\langle u - u_{\frac{1}{2B}}, h_k^{1/2} e_k\rangle d(\beta_t^k - \beta_{t+\frac{1}{2B}}^k)$$

and the bounds

$$\begin{aligned} |u - u_{\frac{1}{2B}}|_2^2(\omega, t) &\leq |u - u_{\frac{1}{2B}}|_2^2(0) e^{-2\nu\lambda_1 t} \\ &\quad + 2\sum_{k\neq 0}\int_0^t e^{-2\nu\lambda_1(t-s)}\langle u - u_{\frac{1}{2B}}, h_k^{1/2} e_k\rangle d(\beta_s^k - \beta_{t+\frac{1}{2B}}^k) \end{aligned}$$

$$\begin{aligned} \int_0^t |\nabla u - \nabla u_{\frac{1}{2B}}|_2^2(\omega, s) ds &\leq \frac{1}{2\nu}|u - u_{\frac{1}{2B}}|_2^2(0) \\ &\quad + \frac{1}{\nu}\sum_{k\neq 0}\int_0^t \langle u - u_{\frac{1}{2B}}, h_k^{1/2} e_k\rangle d(\beta_s^k - \beta_{t+\frac{1}{2B}}^k), \end{aligned}$$

where λ_1 is the smallest eigenvalue of $-\Delta$ with vanishing boundary conditions on the box $[0, 1]^3$ and $h_k = |h_k^{1/2}|^2$. The expectations of these norms are also bounded

$$\begin{aligned} E(|u - \nabla u_{\frac{1}{2B}}|_2^2)(t) &\leq E(|u - \nabla u_{\frac{1}{2B}}|_2^2(0)) e^{-2\nu\lambda_1 t} \\ E(\int_0^t |\nabla u - \nabla u_{\frac{1}{2B}}|_2^2(s) ds) &\leq \frac{1}{2\nu} E(|u - \nabla u_{\frac{1}{2B}}|_2^2(0)) \end{aligned}$$

by the expectations of the initial data of the differences.

The proof of this lemma is analogous to the proof of Lemma 3.2 and can be found in [5].

Remark 3.4. Notice that in the notation of the previous section $|w - w_{\frac{1}{2B}}|_2^2 = |u - u_{\frac{1}{2B}}|_2^2$ because the constant velocity $\mathbf{U}$ cancels out.

4. The a priori estimate of the turbulent solutions

The mechanism of the turbulence production are fast oscillations driving large turbulent noise, that was initially seeded by small white noise, as explained in the previous section. These fast oscillations are generated by the fast constant flow $U = U_1$, where we have dropped the subscript 1, and the flow is rotating with amplitude A and angular velocity Ω. The frequency of these oscillations increases with U and $A\Omega$. The bigger U and $A\Omega$ are the more efficient this turbulence production mechanism becomes.

In this section we will establish an a priori estimate on the norm of the turbulent solution that allows us to extend the local existence and uniqueness to the whole real time axis. Thus the a priori estimates suffices to give global existence and uniqueness. We recall the oscillatory kernal (2.7) from section 2,

$$\sum_{k\neq 0} h_k^{1/2} \int_0^t e^{-(4\pi^2|k|^2+2\pi i U_1 k_1)(t-s)-2\pi i A(k_2,k_3)[\sin(\Omega t+\theta)-\sin(\Omega s+\theta)]} d\beta_s^k e_k(x)$$

The imaginary part of the argument of the exponential creates oscillations and as U_1 and $A\Omega$ become larger these oscillations become faster. We take advantage of this mechanism to produce the a priori estimates

Next lemma plays a key role in the proof of the useful estimate of the turbulent solution. It is a version of the Riemann–Lebesgue Lemma, which captures the averaging effect (mixing) of the oscillations.

Lemma 4.1. *Let the Fourier transform in time be*

$$\tilde{w} = \int_0^T w(s) e^{-2\pi i(k_1 U + A(k_2,k_3)\Omega)s} ds\,,$$

where $A(k_2,k_3) = A\sqrt{k_2^2+k_3^2}$ *and* $w = w(k,t)$, $k = (k_1,k_2,k_3)$ *is a vector with three components. If* T *is an even integer multiple of* $\frac{1}{k_1U+A(k_2,k_3)\Omega}$, *then*

$$\tilde{w} = \widetilde{\eth w}\,,$$

where

$$\eth w = \frac{1}{2}\left(w(s) - w\left(s+\frac{1}{2[k_1U+A(k_2,k_3)\Omega]}\right)\right) = \frac{1}{2}\int_{s+\frac{1}{2[k_1U+A(k_2,k_3)\Omega]}}^{s} \frac{\partial w}{\partial r} dr$$

and $\eth w$ *satisfies the estimate*

$$|\eth w| \leq \frac{1}{4|k_1U+A(k_2,k_3)\Omega|} \operatorname{ess\,sup}_{[s,s+\frac{1}{2(k_1U_1+A(k_2,k_3)\Omega)}]} \left|\frac{\partial w}{\partial s}\right|$$

Proof. The proof is similar to the proof of the Riemann–Lebesgue lemma for the Fourier transform in time, let $B(k) = k_1U + A(k_2,k_3)\Omega$,

$$\begin{aligned}
\tilde{w}(k) &= \int_0^T w(s)e^{-2\pi i B s} ds \\
&= -\int_0^T w(s) e^{-2\pi i B(s-\frac{1}{2B})} ds \\
&= -\int_0^T w\left(s+\frac{1}{2B}\right) e^{-2\pi i B s} ds\,,
\end{aligned}$$

where we have used in the last step that w is a periodic function on the interval $[0,T]$. Taking the average of the first and the last expression we get

$$\tilde{w} = \frac{1}{2}\int_0^T \left(w(s) - w\left(s+\frac{1}{2B}\right)\right) e^{-2\pi i B s} ds = \widetilde{\eth w}$$

Now

$$\begin{aligned}
|\eth w| &= \frac{1}{2}|(w(s) - w(s+\frac{1}{2B}))| \\
&\leq \frac{1}{2}\int_s^{s+\frac{1}{2B}} |\frac{\partial w}{\partial r}|dr \\
&\leq \frac{1}{4|B|}\text{ess sup}_{[s,s+\frac{1}{2B}]}|\frac{\partial w}{\partial s}|
\end{aligned}$$

by the mean-value theorem. □

Corollary 4.2. *If T is not an even integer multiple of $\frac{1}{B(k)} = \frac{1}{k_1U+A(k_2,k_3)\Omega}$, then*

$$\tilde{w} = \widetilde{\eth w} - \frac{1}{2}\int_{-\frac{1}{2B}}^0 w(s+\frac{1}{2B})e^{-2\pi iBs}ds + \frac{1}{2}\int_{T-\frac{1}{2B}}^T w(s+\frac{1}{2B})e^{-2\pi iBs}ds\,,$$

where $\tilde{w}$ satisfies the estimate

$$|\tilde{w}| \leq |\widetilde{\eth w}| + \frac{1}{|B|}\text{ess sup}_{[-\frac{1}{2B},0]\cap[T-\frac{1}{2B},T]}|w(s+\frac{1}{2B})|$$

Proof. The proof is the same as of the Lemma except for the step

$$\begin{aligned}
\tilde{w}(k) &= \int_0^T w(s)e^{-2\pi iBs}ds \\
&= -\int_0^T w(s)e^{-2\pi iB(s-\frac{1}{2B})}ds \\
&= -\int_0^T w(s+\frac{1}{2B})e^{-2\pi iBs}ds \\
&\quad -\int_{-\frac{1}{2B}}^0 w(s+\frac{1}{2B})e^{-2\pi iBs}ds + \int_{T-\frac{1}{2B}}^T w(s+\frac{1}{2B})e^{-2\pi iBs}ds
\end{aligned}$$

□

The lemma allows us to estimate the Fourier transform (in t) of w in terms of the time derivative of w, with a gain of $(k_1U + A(k_2,k_3)\Omega)^{-1}$. Below we will use it in an estimate showing that the limit of $\eth w$ is zero, when $|B(k)| = |(k_1U + A(k_2,k_3)\Omega)| \to \infty$.

Lemma 4.3. *The integral*

$$\int_0^t (2\pi|k|)^p e^{-(4\pi^2\nu|k|^2+2\pi i[B(k)(t-s)+g])}ds\,,$$

where $B(k) = k_1U + A(k_2,k_3)\Omega$, is bounded by

$$(2\pi)^p\int_0^t |k|^p e^{-4\pi^2\nu|k|^2(t-s)}ds \leq C\ t^{1-\frac{p}{2}}$$

for $0 \leq p < 2$, where C is a constant. In particular,

$$\int_{t-\delta}^t (2\pi|k|)^p e^{-(4\pi^2\nu|k|^2+2\pi i[B(k)(t-s)+g])}ds \leq C\ \delta^{1-\frac{p}{2}}.$$

Proof. We estimate the integral

$$\begin{aligned}\int_0^t |k|^p e^{-4\pi^2\nu|k|^2(t-s)} ds &= \int_0^t |k|^p e^{-4\pi^2\nu|k|^2 r} dr \\ &\leq (\frac{p}{4\pi^2})^{\frac{p}{2}} e^{-p} \int_0^t r^{-\frac{p}{2}} dr \\ &= Ct^{1-\frac{p}{2}},\end{aligned}$$

where

$$k = \frac{1}{2\pi}\sqrt{\frac{p}{r}}$$

is the value of k, where the integrand achieves its maximum. □

The rotation can resonate with the uniform (linear) flow due to the nonlinearities in the Navier–Stokes equation. The following lemma restricts the values of velocity coefficients so that no resonance occurs.

Lemma 4.4. *Suppose that for $k_1 < 0$ and $\frac{\sqrt{k_2^2+k_3^2}}{|k_1|} \neq 0$ or ∞, the constants U, A and Ω satisfy the non-resonance condition*

$$\left|\frac{U}{A\Omega} + \frac{\sqrt{k_2^2+k_3^2}}{k_1}\right| \geq \frac{C}{|k_1|^r}, \tag{4.1}$$

where C is a constant and $0 < r < 1$; then for all $k = (k_1, k_2, k_2) \neq 0$,

$$|Uk_1 + A\Omega\sqrt{k_2^2+k_3^2}| \neq 0 \tag{4.2}$$

and

$$\lim_{|k|\to\infty} |Uk_1 + A\Omega\sqrt{k_2^2+k_3^2}| = \infty. \tag{4.3}$$

Moreover,

$$|Uk_1 + A\Omega\sqrt{k_2^2+k_3^2}| \geq B = \min(U, A\Omega, CA\Omega). \tag{4.4}$$

Proof. If $k_1 > 1$, then

$$|Uk_1 + A\Omega\sqrt{k_2^2+k_3^2}| = U|k_1| + A\Omega\sqrt{k_2^2+k_3^2} > 0$$

so (4.2) and (4.3) hold. If $k_1 < 0$, then by (4.1)

$$|Uk_1 + A\Omega\sqrt{k_2^2+k_3^2}| \geq C\ \Omega A|k_1|^{1-r} > 0$$

and

$$\lim_{|k|\to\infty} |Uk_1 + A\Omega\sqrt{k_2^2+k_3^2}| \geq C\ \Omega A \lim_{|k_1|\to\infty} |k_1|^{1-r} = \infty$$

if $|k_1| \to \infty$. If on the other hand $|k_1| < \infty$, when $|k| \to \infty$ then (4.3) also holds. When $k_1 = 0$, (4.2) and (4.3) are obvious and also if $k_2 = k_3 = 0$.

The lower bound (4.4) is read of

$$|Uk_1 + A\Omega\sqrt{k_2^2+k_3^2}|,$$

when $k_1 \geq 1$. Then it is either U or $A\Omega$. When $k_1 = 0$ then it is $A\Omega$ and by (4.1), when $k_1 \leq -1$ it greater than or equal $CA\Omega$. $\square$

The next question to ask is in which space do the turbulent solutions live? This was pointed out by Onsager in 1945 [23]. He pointed out that if the solutions satisfy the Kolmogorov scaling down to the smallest scales, they must be Hölder continuous function with Hölder exponent 1/3. In three dimensions this means that they live in the Sobolev space $H^{\frac{11}{6}+\epsilon}$ based on $L^2(\mathbb{T}^3)$.

If $\frac{q}{p}$ is a rational number let $\frac{q}{p}^+$ denote any real number $s > \frac{q}{p}$.

Theorem 4.5. *Let the velocity $U = U_1$ of the mean flow and the product $A\Omega$ of the amplitude A and the frequency Ω of the rotation be sufficiently large, in the uniform rotating flow (2.1), with U, $A\Omega$ also satisfying the non-resonance conditions (4.1). Then the solution of the integral equation (2.5) is uniformly bounded in $\mathcal{L}^2_{\frac{11}{6}^+}$,*

$$\text{ess sup}_{t\in[0,\infty)} E(\|u\|^2_{\frac{11}{6}^+})(t) \leq (1-C(\frac{1}{B^2}+\delta^{\frac{1}{6}^-}))^{-1}\left[\sum_{k\neq 0}\frac{3(1+(2\pi|k|)^{\frac{11}{3}^+})}{8\pi^2\nu|k|^2}h_k + \frac{C'}{B}\right], \tag{4.5}$$

where $B = min(|U|, A\Omega, CA\Omega)$ is large, δ small and C and C' are constants.

Corollary 4.6. Onsager's Conjecture *The solutions of the integral equation (2.5) are Hölder continuous with exponent 1/3.*

Remark 4.7. The estimate (4.5) provides the answer to the question we posed in Section 2 how fast the coefficients $h_k^{1/2}$ had to decay in Fourier space. They have to decay sufficiently fast for the expectation of the $H^{\frac{11}{6}^+} = W^{(\frac{11}{6}^+,\,2)}$ Sobolev norm of the initial function u_o, to be finite. This expectation appear on the right hand side of (4.5). In other words the $\mathcal{L}^2_{\frac{11}{6}^+}$ norm of the initial function u_o has to be finite.

We now give an outline of the proof of the theorem. More details can be found in [5].

Outline of Proof: We write the integral equation (2.5) in the form

$$\begin{aligned} u(x,t) &= \sum_{k\neq 0}[h_k^{1/2}A_t^k - \int_0^t e^{-(\{4\pi^2\nu|k|^2+2\pi i[k_1U_1+A(k_2,k_3)\Omega]\}(t-s)+2\pi i g(k,t,s))} \\ &\quad \times(\widehat{u\cdot\nabla u} - \nabla\Delta^{-1}\widehat{(\mathrm{tr}(\nabla u)^2)})(k,s)ds]e_k(x)\,, \end{aligned}$$

where $e_k = e^{2\pi i k\cdot x}$ are the Fourier components and the A_t^k are the oscillatory Ornstein–Uhlenbeck-type processes (2.8) and $\mathrm{tr}(\nabla u)^2$ denotes the trace of the matrix $(\nabla u)^2$. The Fourier transform of the term $\nabla\Delta^{-1}(\mathrm{tr}(\nabla u)^2)$ is just

$\frac{-ik}{2\pi|k|^2}\widehat{\mathrm{tr}(\nabla u)^2}$ and we will write the integral equation in the form

$$\begin{aligned} u(x,t) &= \sum_{k\neq 0}[h_k^{1/2}A_t^k - \int_0^t e^{-[4\pi^2\nu|k|^2+2\pi iB(k)](t-s)-2\pi ig(k,t,s)} \\ &\quad \times(\widehat{u\cdot\nabla u}+\frac{ik}{2\pi|k|^2}(\widehat{\mathrm{tr}(\nabla u)^2}))(k,s)ds]e_k(x)\,, \end{aligned} \tag{4.6}$$

where $B(k) = Uk_1 + A(k_2,k_3)\Omega$, from here on with

$$g(k,t,s) = A(k_2,k_3)[\Omega(t-s) - (\sin(\Omega t+\theta) - \sin(\Omega s+\theta))]$$

We will also assume the trivial non-resonance conditions that A and Ω are sufficiently incommensurate for the rest of the paper.

We split the t integral into the integral from 0 to $t-\delta$, where δ is a small number, and the integral from $t-\delta$ to t. This is done to first avoid the singularities of the spatial derivatives of the heat kernal at $s = t$ and then to deal with these singularities in the latter integral. Now the first estimate is relatively straightforward. The L^2 norm of

$$\sum_{k\neq 0}\int_{t-\delta}^t e^{-\{(4\pi^2\nu|k|^2+2\pi iB)(t-s)+2\pi ig(k,t,s)\}}(-\widehat{u\cdot\nabla u})ds\ e_k$$

is

$$\begin{aligned} &\sum_{k\neq 0}|\int_{t-\delta}^t e^{-\{(4\pi^2\nu|k|^2+2\pi iB)(t-s)+2\pi ig(k,t,s)\}}(-\widehat{u\cdot\nabla u})ds|^2 \\ \leq\ & \delta\sum_{k\neq 0}\int_{t-\delta}^t|\widehat{u\cdot\nabla u}|_2^2(k)ds \leq\ \delta\ \int_{t-\delta}^t|u\cdot\nabla u|_2^2ds \\ \leq\ & \delta\ \text{ess sup}_{[t-\delta,t]}|u|_\infty^2\int_{t-\delta}^t|\nabla u|_2^2ds \\ \leq\ & (\frac{\delta}{\nu}\int_{t-\delta}^t\langle u, h_k^{1/2}e_k\rangle d\beta_s^k + \frac{\delta^2}{2\nu}\sum_{k\neq 0}h_k)\ \text{ess sup}_{[t-\delta,t]}\|u\|_{\frac{3}{2}^+}(s) \end{aligned}$$

since by the Gagliardo–Nirenberg inequalities

$$|u|_\infty \leq C\|u\|_{\frac{3}{2}^+}\,,$$

where δ is independent of U_1 and C is a constant, and by the a priori estimate in Lemma 3.2. Similarly, the L^2 norm of

$$\sum_{k\neq 0}\int_{t-\delta}^t e^{-\{(4\pi^2\nu|k|^2+2\pi iB)(t-s)+2\pi ig(k,t,s)\}}(\frac{ik}{2\pi|k|^2}(\widehat{\mathrm{tr}(\nabla u)^2}))ds\ e_k$$

is

$$
\begin{aligned}
& \sum_{k\neq 0} | \int_{t-\delta}^{t} e^{-\{(4\pi^2\nu|k|^2+2\pi iB)(t-s)+2\pi i g(k,t,s)\}} (\frac{ik}{2\pi|k|^2}(\widehat{\mathrm{tr}(\nabla u)^2})) ds|^2 \\
\leq\ & \delta \sum_{k\neq 0} \int_{t-\delta}^{t} |(\frac{ik}{2\pi|k|^2}(\widehat{\mathrm{tr}(\nabla u)^2}))|_2^2(k) ds \\
\leq\ & \frac{\delta}{2\pi} \text{ ess sup}_{[t-\delta,t]} |u|_\infty^2 \int_{t-\delta}^{t} |\nabla u|_2^2 ds \\
\leq\ & (\frac{\delta}{2\pi\nu} \int_{t-\delta}^{t} \langle u, h_k^{1/2} e_k\rangle d\beta_s^k + \frac{\delta^2}{4\pi\nu} \sum_{k\neq 0} h_k) \text{ ess sup}_{[t-\delta,t]} \|u\|^2_{\frac{3}{2}+}(s)
\end{aligned}
$$

The other integrals are estimated by use of Lemma 4.1. The integral

$$
\int_0^{t-\delta} e^{-\{(4\pi^2\nu|k|^2+2\pi iB)(t-s)+2\pi i g(k,t,s)\}} \widehat{u\cdot\nabla u} ds
$$

can be estimated by Lemma 4.1, when $t-\delta$ is an even integer multiple of $\frac{1}{B}$, we get that

$$
\begin{aligned}
& \int_0^{t-\delta} e^{-\{4\pi^2\nu|k|^2+2\pi iB)(t-s)+2\pi i g(k,t,s)\}} \widehat{u\cdot\nabla u}(s) ds \\
= \ & \frac{1}{2} \int_0^{t-\delta} [e^{-\{4\pi^2\nu|k|^2(t-s)+2\pi i g(k,t,s)\}} \widehat{u\cdot\nabla u}(s) \\
& -e^{-\{4\pi^2\nu|k|^2+2\pi iB)(t-(s+\frac{1}{2B}))+2\pi i g(k,t,s+\frac{1}{2B})\}} \widehat{u\cdot\nabla u}(s+\frac{1}{2B})] e^{-2\pi iB(t-s)} ds \\
= \ & \frac{1}{2} \int_0^{t-\delta} [(e^{-\{4\pi^2\nu|k|^2(t-s)+2\pi i g(k,t,s)\}} \\
& -e^{-\{4\pi^2\nu|k|^2(t-(s+\frac{1}{2B}))+2\pi i g(k,t,s+\frac{1}{2B})\}}) \widehat{u\cdot\nabla u}(s)] e^{-2\pi iB(t-s)} ds \\
& +\frac{1}{2} \int_0^{t-\delta} \{e^{-(4\pi^2\nu|k|^2(t-(s+\frac{1}{2B}))+2\pi i g(k,t,s+\frac{1}{2B})\}} ([\hat{u}(s) - \widehat{u(s+\frac{1}{2B})}] * \widehat{\nabla u(s)} \\
& +\widehat{u(s+\frac{1}{2B})} * [\widehat{\nabla u(s)} - \widehat{\nabla u(s+\frac{1}{2B})}]) \} e^{-2\pi iB(t-s)} ds
\end{aligned}
$$

The first term in the last line above is estimated by Schwartzes inequality

$$
\begin{aligned}
& |\int_0^{t-\delta}[(e^{-\{4\pi^2\nu|k|^2(t-s)+2\pi i g(k,t,s)\}} \\
& -e^{-\{4\pi^2\nu|k|^2(t-(s+\frac{1}{2B}))+2\pi i g(k,t,s+\frac{1}{2B})\}})\widehat{u\cdot\nabla u(s)}]e^{-2\pi iB(t-s)}ds|^2 \\
\leq\ & \int_0^{t-\delta}|e^{-\{2\pi^2\nu|k|^2(t-s)+2\pi i g(k,t,s)\}} \\
& -e^{-\{2\pi^2\nu|k|^2(t-(s+\frac{1}{B}))+2\pi i g(k,t,s+\frac{1}{2B})\}}|^2|u|_2^2(s)\ ds\ \int_0^{t-\delta}e^{-4\pi^2\nu|k|^2(t-s)}|\nabla u|_2^2(s)ds \\
\leq\ & e^{-2\pi^2\nu|k|\delta}\times \\
& \int_0^{t-\delta}|e^{-\{2\pi^2\nu|k|^2(t-s)+2\pi i g(k,t,s)\}}-e^{-\{2\pi^2\nu|k|^2(t-(s+\frac{1}{B}))+2\pi i g(k,t,s+\frac{1}{2B})\}}|^2ds \\
& \times\int_0^{t-\delta}e^{-2\pi^2\nu|k|^2(t-s)}|\nabla u|_2^2(s)ds\ \text{ess sup}_{s\in[0,t-\delta]}|u|_2^2(s) \\
\leq\ & \frac{Ce^{-2\pi^2\nu|k|\delta}}{B^2}\int_0^{t-\delta}e^{-2\pi\nu|k|^2(t-s)}|\nabla u|_2^2(s)ds\ \text{ess sup}_{s\in[0,t-\delta]}|u|_2^2(s)
\end{aligned}
$$

by Lemma 4.1. Similarly the second term is estimated by

$$
\begin{aligned}
& |\int_0^{t-\delta}e^{-(4\pi^2\nu|k|^2(t-(s+\frac{1}{2B}))+2\pi i g(k,t,s+\frac{1}{2B})\}}([\hat{u}(s)-\widehat{u(s+\frac{1}{2B}})] \\
& *\widehat{\nabla u(s)}e^{-2\pi iB(t-s)}ds|^2 \\
\leq\ & e^{-4\pi^2\nu|k|^2(\delta-\frac{1}{2B})}\int_0^{t-\delta}|u(s)-u(s+\frac{1}{2B})|_2^2ds\ \int_0^{t-\delta}e^{-4\pi^2\nu|k|^2s}|\nabla u|_2^2(s)ds
\end{aligned}
$$

using the Cauchy-Schwartz inequality both on the convolution and the time-integral, and the third term is estimated by

$$
\begin{aligned}
& |\int_0^{t-\delta}e^{-(4\pi^2\nu|k|^2(t-(s+\frac{1}{2B}))+2\pi i g(k,t,s+\frac{1}{2B})\}}(\widehat{u(s+\frac{1}{2B}})) \\
& *[\widehat{\nabla u(s)}-\widehat{\nabla u(s+\frac{1}{2B}})]e^{-2\pi iB(t-s)}ds|^2 \\
\leq\ & \frac{e^{-8\pi^2\nu|k|(\delta-\frac{1}{2B})}}{8\nu\pi^2|k|^2}\int_0^{t-\delta}|\nabla u(s)-\nabla u(s+\frac{1}{2B})|_2^2ds\ \text{ess sup}_{s\in[0,t]}|u|_2^2(s+\frac{1}{2B})
\end{aligned}
$$

Now the terms

$$H=\int_0^{t-\delta}|u(s)-u(s+\frac{1}{2B})|_2^2ds\ \int_0^{t-\delta}e^{-4\pi^2\nu|k|^2s}|\nabla u|_2^2(s)ds$$

and

$$K=\int_0^{t-\delta}|\nabla u(s)-\nabla u(s+\frac{1}{2B})|_2^2ds\ \text{ess sup}_{s\in[0,t]}|u|_2^2(s+\frac{1}{2B})$$

are estimated by use of Lemma 3.3 and Lemma 4.8. Thus the a priori bounds on the L^2 norms of u and ∇u and their differences in those two lemmas and in

Lemmas 3.2 and 4.9 give the inequality

$$
\begin{aligned}
& | \int_0^{t-\delta} e^{-\{(4\pi^2\nu|k|^2+2\pi iB)(t-s)+2\pi i g(k,t,s)\}} \widehat{u\cdot\nabla u}(s)ds|^2 \\
\leq \quad & Ce^{-4\pi^2\nu|k|(\delta-\frac{1}{2B})} \text{ ess sup}_{s\in[0,t]}(\frac{C}{B^2}+H+K+d(k)),
\end{aligned}
$$

where the terms H and K are estimated in Lemma 4.10 and the expectation of $d(k)$ vanishes.

Now consider the pressure term. By use of Lemma 4.1, we get that

$$
\begin{aligned}
& \int_0^{t-\delta} e^{-\{4\pi^2\nu|k|^2+2\pi iB)(t-s)+2\pi i g(k,t,s)\}} \frac{ik}{2\pi|k|^2}\widehat{\text{tr}(\nabla u)^2}ds \\
= \quad & \frac{1}{2}\int_0^{t-\delta} \{e^{-\{4\pi^2\nu|k|^2(t-s)+2\pi i g(k,t,s)\}} \frac{ik}{2\pi|k|^2}\widehat{\text{tr}(\nabla u)^2}(s) \\
& -e^{-\{4\pi^2\nu|k|^2(t-(s+\frac{1}{2B}))+2\pi i g(k,t,s+\frac{1}{2B})\}} \frac{ik}{2\pi|k|^2}\widehat{\text{tr}(\nabla u)^2}(s+\frac{1}{2B})\}e^{-2\pi iB(t-s)}ds \\
= \quad & \frac{1}{2}\int_0^{t-\delta} \{e^{-\{4\pi^2\nu|k|^2(t-s)+2\pi i g(k,t,s)\}} \\
& -e^{-\{4\pi^2\nu|k|^2(t-(s+\frac{1}{2B}))+2\pi i g(k,t,s+\frac{1}{2B})\}}\} \frac{ik}{2\pi|k|^2}\widehat{\text{tr}(\nabla u)^2}(s)ds \\
& +\frac{1}{2}\int_0^{t-\delta} e^{-\{4\pi^2\nu|k|^2(t-(s+\frac{1}{2B}))+2\pi i g(k,t,s+\frac{1}{2B})\}} \\
& \times\frac{ik}{2\pi|k|^2}\text{tr}[(\widehat{\nabla u(s)}-\widehat{\nabla u(s+\frac{1}{2B})})*(\widehat{\nabla u(s)}+\widehat{\nabla u(s+\frac{1}{2B})})]e^{-2\pi iB(t-s)}ds
\end{aligned}
$$

The first term in the last expression above is estimated as

$$
\begin{aligned}
& |\int_0^{t-\delta} \{e^{-\{4\pi^2\nu|k|^2(t-s)+2\pi i g(k,t,s)\}} \\
& -e^{-\{4\pi^2\nu|k|^2(t-(s+\frac{1}{2B}))+2\pi i g(k,t,s+\frac{1}{2B})\}}\} \frac{ik}{2\pi|k|^2}\widehat{\text{tr}(\nabla u)^2}(s)ds|^2 \\
\leq \quad & \int_0^{t-\delta} |e^{-\{2\pi^2\nu|k|^2(t-s)+2\pi i g(k,t,s)\}} \\
& -e^{-\{2\pi^2\nu|k|^2(t-(s+\frac{1}{B}))+2\pi i g(k,t,s+\frac{1}{2B})\}}|^2|u|_2^2(s)ds \quad \int_0^{t-\delta} e^{-4\pi\nu|k|^2(t-s)}|\nabla u|_2^2(s)ds \\
\leq \quad & e^{-2\pi^2\nu|k|\delta}\int_0^{t-\delta} |e^{-\{2\pi^2\nu|k|^2(t-s)+2\pi i g(k,t,s)\}} \\
& -e^{-\{2\pi^2\nu|k|^2(t-(s+\frac{1}{B}))+2\pi i g(k,t,s+\frac{1}{2B})\}}|^2 ds\int_0^{t-\delta} e^{-2\pi\nu|k|^2(t-s)}|\nabla u|_2^2(s)ds \\
& \times\text{ess sup}_{s\in[0,t-\delta]}|u|_2^2(s) \\
\leq \quad & \frac{Ce^{-2\pi^2\nu|k|\delta}}{B^2}\int_0^{t-\delta} e^{-2\pi\nu|k|^2(t-s)}|\nabla u|_2^2(s)ds \text{ ess sup}_{s\in[0,t-\delta]}|u|_2^2(s)
\end{aligned}
$$

The second term is estimated by

$$
\begin{aligned}
& |\int_0^{t-\delta} e^{-\{4\pi^2\nu|k|^2(t-(s+\frac{1}{2B}))+2\pi i g(k,t,s+\frac{1}{2B})\}} \\
\times \quad & \frac{ik}{2\pi|k|^2}\mathrm{tr}[(\widehat{\nabla u(s)}-\widehat{\nabla u(s+\frac{1}{2B}}))*(\widehat{\nabla u(s)}+\widehat{\nabla u(s+\frac{1}{2B}}))]e^{-2\pi i B(t-s)}ds|^2 \\
\leq \quad & e^{-4\pi^2\nu|k|(\delta-\frac{1}{2B})}\int_0^{t-\delta}|\nabla u(s)-\nabla u(s+\frac{1}{2B})|_2^2 ds\int_0^{t-(\delta-\frac{1}{2B})} e^{-4\pi^2\nu|k|^2 s}|\nabla u|_2^2(s)ds
\end{aligned}
$$

Thus

$$
\begin{aligned}
& |\int_0^{t-\delta} e^{-\{(4\pi^2\nu|k|^2+2\pi i B)(t-s)+2\pi i g(k,t,s)\}}\frac{ik}{2\pi|k|^2}\widehat{\mathrm{tr}(\nabla u)^2}ds|^2 \\
& \leq Ce^{-4\pi^2\nu|k|(\delta-\frac{1}{2B})}\ \text{ess sup}_{s\in[0,t]}\big(\frac{C}{|B(k)|^2}+L+d(k)\big)\,,
\end{aligned}
$$

where the expectation of $d(k)$ vanishes and the term

$$
L=\int_0^{t-\delta}|\nabla u(s)-\nabla u(s+\frac{1}{2B})|_2^2 ds\ \int_0^{t-(\delta-\frac{1}{2B})} e^{-4\pi^2\nu|k|^2 s}|\nabla u|_2^2(s)ds
$$

is estimated in Lemma 4.10, again by the a priori bounds on the L^2 norms of u and ∇u and their differences in Lemma 3.2 and Lemma 4.9, and Lemmas 3.3 and 4.8.

When $t-\delta$ is not an even integer multiple of $\frac{1}{B(k)}$ we get the additional terms in Corollary 4.2. However these are estimated exactly as the integrals from $t-\delta$ to t and simply add another term multiplied by δ^2 if we choose $\frac{1}{|B|}=\sup_{k\neq 0}\frac{1}{|B(k)|}<\delta$.

Now we assemble the estimates. Up to terms that vanish, when the expectation is taken, the L^2 norm of u is bounded by

$$
\begin{aligned}
|u|_2^2 \quad \leq \quad & 3\sum_{k\neq 0} h_k|A_t^k|^2 \\
& +3\sum_{k\neq 0}(|\int_0^{t-\delta} e^{-(\{4\pi^2\nu|k|^2+2\pi i[k_1U_1+A(k_2,k_3)\Omega]\}(t-s)+2\pi i g(k,t,s))} \\
& \times(\widehat{u\cdot\nabla u}-\nabla\Delta^{-1}\widehat{(\mathrm{tr}(\nabla u)^2)})(k,s)ds|^2+\delta^2 C\text{ess sup}_{s\in[t-\delta,t]}\|u\|^2_{\frac{11}{6}+} \\
\leq \quad & 3\sum_{k\neq 0} h_k|A_t^k|^2 \\
& +\sum_{k\neq 0} e^{-4\pi^2\nu|k|(\delta-\frac{1}{2B})}[\frac{C'}{|B(k)|^2}+H+K+L](s)+\delta^2 C\text{ess sup}_{s\in[t-\delta,t]}\|u\|^2_{\frac{11}{6}+} \\
\leq \quad & 3\sum_{k\neq 0} h_k|A_t^k|^2+C(\frac{1}{B^2}+\delta^2)\text{ess sup}_{s\in[t-\delta,t]}\|u\|^2_{\frac{11}{6}+}+\frac{C'}{B}
\end{aligned}
$$

by Lemma 4.10.

We now act on the integral equation (4.6) with the operator $\nabla^{(11/6)^+}$, to estimate the derivative $\nabla^{(11/6)^+}u$

$$\begin{aligned}\nabla^{(11/6)^+}u(x,t) &= \sum_{k\neq 0}[(2\pi i|k|)^{(11/6)^+}h_k^{1/2}A_t^k \\ &\quad - \int_0^t (2\pi i|k|)^{(11/6)^+}e^{-[4\pi^2\nu|k|^2+2\pi iB(k)](t-s)-2\pi ig(k,t,s)} \\ &\quad \times(\widehat{u\cdot\nabla u}+\frac{ik}{2\pi|k|^2}(\widehat{\mathrm{tr}(\nabla u)^2}))(k,s)ds]e_k(x)\,,\end{aligned}$$

where $B(k)$ and $g(k,t,s)$ are as in (4.6). An estimate similar to Equation (4.7) gives

$$\begin{aligned}|\nabla^{(11/6)^+}u|_2^2 \leq\ & 3\sum_{k\neq 0}(2\pi|k|)^{(11/3)^+}h_k|A_t^k|^2 \\ & +3\sum_{k\neq 0}(|\int_0^{t-\delta}|k|^{\frac{11}{6}^+}e^{-(\{4\pi^2\nu|k|^2+2\pi i[k_1U_1+A(k_2,k_3)\Omega]\}(t-s)+2\pi ig(k,t,s))} \\ & \times(\widehat{u\cdot\nabla u}-\nabla\Delta^{-1}\widehat{(\mathrm{tr}(\nabla u)^2)})(k,s)ds|^2 \\ & +\delta^{\frac{1}{6}^-}C\text{ess sup}_{s\in[t-\delta,t]}\|u\|^2_{\frac{11}{6}+} \\ \leq 3\sum_{k\neq 0}(2\pi|k|)^{(11/3)^+}h_k|A_t^k|^2 & \quad + \text{ess sup}_{s\in[0,t-\delta]}[\frac{C'}{B^2}+H+K+L](s) \\ & \quad +\delta^{\frac{1}{6}^-}C\text{ ess sup}_{s\in[t-\delta,t]}\|u\|^2_{\frac{11}{6}+} \qquad (4.7) \\ \leq 3\sum_{k\neq 0}(2\pi|k|)^{(11/3)^+}h_k|A_t^k|^2 & \quad +C(\frac{1}{B^2}+\delta^{\frac{1}{6}^-})\text{ess sup}_{s\in[t-\delta,t]}\|u\|^2_{\frac{11}{6}+}+\frac{C'}{B}\end{aligned}$$

again by Lemma 4.10.

Combining the estimates (4.7) and (4.7) we now get that

$$\|u\|^2_{\frac{11}{6}+}\leq 3\sum_{k\neq 0}(1+(2\pi|k|)^{\frac{11}{3}^+})h_k|A_t^k|^2+C(\frac{1}{B^2}+\delta^{\frac{1}{6}^-})\text{ess sup}_{s\in[t-\delta,t]}\|u\|^2_{\frac{11}{6}+}+\frac{C'}{B}\,,$$

where $\frac{1}{B}$ and δ can be made arbitrarily small. Then taking the expectation we get

$$(1-C(\frac{1}{B^2}+\delta^{\frac{1}{6}^-}))E(\text{ess sup}_{[0,t]}\|u\|^2_{\frac{11}{6}+})\leq 3\sum_{k\neq 0}(1+(2\pi|k|)^{\frac{11}{3}^+})h_kE(|A_t^k|^2)+\frac{C'}{B}$$

and evaluating the last expectation

$$\sum_{k\neq 0}(1+(2\pi|k|)^{\frac{11}{3}^+})h_kE(|A_t^k|^2)=\sum_{k\neq 0}\frac{(1+(2\pi|k|)^{\frac{11}{3}^+})}{8\pi^2\nu|k|^2}h_k$$

gives the estimate (4.5)

$$(1-C(\frac{1}{B^2}+\delta^{\frac{1}{6}^-}))E(\text{ess sup}_{[0,t]}\|u\|^2_{\frac{11}{6}+})\leq 3\sum_{k\neq 0}\frac{(1+(2\pi|k|)^{\frac{11}{3}^+})}{8\pi^2\nu|k|^2}h_k+\frac{C'}{B}$$

By making δ and $\frac{1}{B}$ sufficiently small we conclude that (4.5) holds for all t. *End of Outline of Proof.*

We consider the integral equation

$$\begin{aligned} u(x,t) &= \sum_{k\neq 0}[h_k^{1/2}A_t^k-\int_0^t e^{-[4\pi^2\nu|k|^2+2\pi iB(k)](t-s)-2\pi ig(k,t,s)} \\ &\times(\widehat{u\cdot\nabla u}+\frac{ik}{2\pi|k|^2}(\widehat{\text{tr}(\nabla u)^2}))(k,s)ds]e_k(x)\,, \end{aligned}$$

where $B(k)=Uk_1+A(k_2,k_3)\Omega$

Lemma 4.8. *The initial condition $(u-u_{\frac{1}{2B}})(0)$ satisfies the estimate*

$$|u-u_{\frac{1}{2B}}|_2^2(0)\leq 2\sum_{j\neq 0}|A^j_{\frac{1}{2B(k)}}|^2+\frac{C}{|B(k)|^2}\text{ess sup}_{t\in[0,\frac{1}{2B}]}\|u\|^2_{\frac{11}{6}+}$$

Proof. We use the integral equation

$$\begin{aligned} u-u_{\frac{1}{2B}} &= \sum_{k\neq 0}[h_k^{1/2}(A_t^k-A^k_{t+\frac{1}{2B}}) \\ &-(\int_0^t e^{-[4\pi^2\nu|k|^2+2\pi iB(k)](t-s)-2\pi ig(k,t,s)}\times(\widehat{u\cdot\nabla u}+\frac{ik}{2\pi|k|^2}(\widehat{\text{tr}(\nabla u)^2}))(k,s)ds \\ &-\int_0^{t+\frac{1}{2B}} e^{-[4\pi^2\nu|k|^2+2\pi iB(k)](t+\frac{1}{2B}-s)-2\pi ig(k,t+\frac{1}{2B},s)} \\ &\times(\widehat{u\cdot\nabla u}+\frac{ik}{2\pi|k|^2}(\widehat{\text{tr}(\nabla u)^2}))(k,s)ds]e_k(x)\,, \end{aligned}$$

where $B(k)=Uk_1+A(k_2,k_3)\Omega$. At $t=0$,

$$|u-u_{\frac{1}{2B}}|^2(0)=|u_{\frac{1}{2B}}|^2(0)=2\sum_{j\neq 0}h_j|A^j_{\frac{1}{2B}}|^2++\frac{C}{|B(k)|^2}\text{ess sup}_{t\in[0,\frac{1}{2B}]}\|u\|^2_{\frac{11}{6}+}$$

by the same estimates as above. □

Lemma 4.9. *The identity (3.1) in Lemma 3.2 can be modified for $a>0$*

$$d(e^{\nu at}|u|_2^2)+2\nu e^{\nu at}|\nabla u|_2^2dt=\nu ae^{\nu at}|u|_2^2dt+2e^{\nu at}\sum_{k\neq 0}\langle u,h_k^{1/2}e_k\rangle d\beta_t^k+e^{\nu at}\sum_{k\neq 0}h_kdt \tag{4.8}$$

and produces the estimates

$$
\begin{aligned}
|u|_2^2(t) \quad &\leq \quad |u|_2^2(0)(e^{-\nu a t} + \frac{ae^{-2\nu\lambda_1 t}}{(a-2\lambda_1)}) + 2\sum_{k\neq 0}\int_0^t e^{-\nu a(t-s)}\langle u, h_k^{1/2}e_k\rangle d\beta_s^k \\
&+2\sum_{k\neq 0}\int_0^t e^{-\nu a(t-s)}\int_0^s e^{-2\nu\lambda_1(s-r)}\langle u, h_k^{1/2}e_k\rangle d\beta_r^k ds + \frac{1}{\nu}(\frac{1}{a}+\frac{1}{2\lambda_1})\sum_{k\neq 0} h_k
\end{aligned} \tag{4.9}
$$

and

$$
\begin{aligned}
&\int_0^t e^{-\nu a(t-s)}|\nabla u|_2^2(s)ds \leq \frac{1}{2\nu}(|u|_2^2(0) - |\mathbf{U}|^2)(e^{-\nu a t} + \frac{ae^{-2\nu\lambda_1 t}}{(a-2\lambda_1)}) \\
&+\frac{1}{\nu}\sum_{k\neq 0}\int_0^t e^{-\nu a(t-s)}\langle u, h_k^{1/2}e_k\rangle d\beta_s^k \\
&+\frac{1}{\nu}\sum_{k\neq 0}\int_0^t e^{-\nu a(t-s)}\int_0^s e^{-2\nu\lambda_1(s-r)}\langle u, h_k^{1/2}e_k\rangle d\beta_r^k ds + \frac{1}{2\nu^2}(\frac{1}{a}+\frac{1}{2\lambda_1})\sum_{k\neq 0} h_k\,,
\end{aligned} \tag{4.10}
$$

where λ_1 is the smallest eigenvalue of $-\Delta$ with vanishing boundary conditions on the box $[0,1]^3$ and $h_k = |h_k^{1/2}|^2$.

Proof. We multiply the identity (3.1) in Lemma 3.2 by $e^{\nu a t}$ to get (4.8). Then integration gives the equality

$$
\begin{aligned}
&|u|_2^2(t) + 2\nu\int_0^t e^{-\nu a(t-s)}|\nabla u|_2^2(s)ds = |u|_2^2(0)e^{-\nu a t} + \nu a\int_0^t e^{-\nu a(t-s)}|u|_2^2(s)ds \\
&+2\sum_{k\neq 0}\int_0^t e^{-\nu a(t-s)}\langle u, h_k^{1/2}e_k\rangle d\beta_s^k + \frac{(1-e^{-\nu a(t-s)})}{\nu a}\sum_{k\neq 0} h_k
\end{aligned}
$$

Now substituting the estimate (3.2), from Lemma 3.2, for $|u|_2^2$ on the right hand side gives the two inequalities (4.9) and (4.10) as in Lemma 3.2. □

Lemma 4.10. *The functions H, K, L in the proof of Theorem 4.5 satisfy the estimate*

$$
E(H+K+L) \quad \leq \quad \frac{C}{|B(k)|^2}E(\text{ess sup}_{t\in[0,\frac{1}{2B}]}\|u\|^2_{\frac{11}{6}+}) + \frac{C'}{B}
$$

with $B = min(U, A\Omega, CA\Omega)$.

The proof of the lemma involves long formulas for H, K and L and can be found in [5].

Remark 4.11. Corollary 4.6 is the resolution of a famous question in turbulence: *Is turbulence always caused by the blow-up of the velocity u ?* The answer according to Theorem 4.5 is *no*; the solutions are not singular. However, they are not smooth either, contrary to the belief, stemming from Leray's theory [17], that if solutions are not singular then they are smooth. By Corollary 4.6 the solutions are Hölder continuous with exponent 1/3 in three dimensions. This confirms a conjecture made by Onsager [24] in 1945. In particular the gradient ∇u and vorticity $\nabla \times u$ are not continuous in general.

Remark 4.12. U and $A\Omega$ do not have to be made very large for the estimate (4.5) to be satisfied, because $B(k) \to \infty$ as $|k| \to \infty$. How big U and $A\Omega$ have to be for (4.5) to hold is probably best answered by a numerical simulation.

We can now prove that $\text{ess sup}_{t\in[0,\infty)} \|u(t)\|^2_{\frac{11}{6}+}$ is bounded with probability close to one.

Lemma 4.13. *For all $\epsilon > 0$ there exists an R such that,*

$$\mathbb{P}(\text{ess sup}_{t\in[0,\infty)} \|u(t)\|^2_{\frac{11}{6}+} < R) > 1 - \epsilon$$

Proof. By Chebychev's inequality and the estimate (4.5) we get that

$$\mathbb{P}(\text{ess sup}_{t\in[0,\infty)} \|u(t)\|^2_{\frac{11}{6}+} \geq R) < \frac{C}{R} < \epsilon$$

for R sufficiently large. □

5. Global existence of turbulent solutions

In this section we prove the existence of the turbulent solutions of the initial value problem (2.3). The following theorem states the existence of turbulent solutions in three dimensions. First we write the initial value problem (2.3) as the integral equation (5.1),

$$u(x,t) = u_o(x,t) - \int_0^t K(t-s) * [u \cdot \nabla u - \nabla \Delta^{-1} \text{tr}(\nabla u)^2] ds \tag{5.1}$$

Here K is the oscillatory heat kernal (2.6) and

$$u_o(x,t) = \sum_{k\neq 0} h_k^{1/2} A_t^k e_k(x)$$

the A_t^ks being the oscillatory Ornstein–Uhlenbeck-type processes from Equation (2.7).

Theorem 5.1. *If the uniform flow U and product of the amplitude and frequency $A\Omega$, of the rotation, are sufficiently large, $B = min(|U|, A\Omega, CA\Omega)$, δ is small and the non-resonance conditions (4.1) are satisfied, so that the a priori bound (4.5) holds, then the integral equation (5.1) has unique global solution $u(x,t)$ in the space $C([0,\infty); L^2(\Omega, \mathcal{F}, \mathbb{P}; H^{\frac{11}{6}^+}))$, u is adapted to the filtration generated by the stochastic process*

$$u_o(x,t) = \sum_{k\neq 0} h_k^{1/2} A_t^k e_k$$

and

$$E(\int_0^t \|u\|^2_{\frac{11}{6}+} ds) \leq (1 - C(\frac{1}{B^2} + \delta^{\frac{1}{6}^-}))^{-1} \left[\sum_{k\neq 0} \frac{3(1+(2\pi|k|)^{\frac{11}{3}^+})}{8\pi^2 \nu |k|^2} h_k + \frac{C'}{B} \right] \ t$$

This theorem is a standard application of the contraction mapping principle to prove global existence and uniqueness. Then the unique local solution is extended to the whole positive time axis by use of the a priori bound (4.5). A detailed proof can be found in [5].

We now add the initial condition $u(x,0) = u^0(x)$, with mean zero, to the integral equation (5.1).

Theorem 5.2. *If the uniform flow U and the product of the amplitude $A\Omega$ and frequency of the rotation, $B = min(|U|, A\Omega, CA\Omega)$, are sufficiently large, δ small, and the non-resonance conditions (4.1) are satisfied, so that the a priori bound (4.5) holds, then the integral equation*

$$u(x,t) = K(t) * u^0(x) + u_o(x,t) - \int_0^t K(t-s) * (u \cdot \nabla u - \nabla \Delta^{-1} (\nabla u)^2)\, ds\,, \quad (5.2)$$

where K is the oscillating kernal in (2.6), has unique global solution $u(x,t)$ in the space $C([0,\infty); L^2(\Omega, \mathcal{F}, \mathbb{P}; H^{\frac{11}{6}^+}))$, u is adapted to the filtration generated by the stochastic process

$$u_o(x,t) = \sum_{k \neq 0} h_k^{1/2} A_t^k e_k$$

and

$$E\left(\int_0^t \|u\|^2_{\frac{11}{6}^+} ds\right) \leq \left(1 - C\left(\frac{1}{B^2} + \delta^{\frac{1}{6}^-}\right)\right)^{-1} \left[\sum_{k \neq 0} \frac{(1 + (2\pi|k|)^{\frac{11}{3}^+})}{2\pi^2 \nu |k|^2} h_k + \frac{C'}{B}\right]\ t.$$

The proof of the theorem is exactly the same as the proof of Theorem 5.1 once the a priori bound (4.5) is established. A proof can be found in [5].

Corollary 5.3. *For any initial data $u^0 \in \dot{L}^2(\mathbb{T}^3)$, the L^2 space with mean zero, and any $t_0 > 0$, there exists a mean flow U, an amplitude and angular velocity $A\Omega$, and δ small, such that (5.2) has a unique solution in $C([t_0,\infty); L^2(\Omega, \mathcal{F}, \mathbb{P}; H^{\frac{11}{6}^+}))$.*

Proof. For $t > 0$, $K(t) * u^0(x)$ is smooth. Now apply Theorem 5.2 $\square$

Next we prove a Gronwall estimate that will be used in later sections.

Lemma 5.4. *Let u be a solution of (5.1) with an initial function $u_o(x,t) = \sum_{k \neq 0} h_k^{1/2} A_t^k e_k$ and initial condition $u^0(x)$ and y a solution of*

$$y_t + \mathbf{U} \cdot \nabla y = \nu \Delta y - y \cdot \nabla y + \nabla \Delta^{-1} \mathrm{tr}(\nabla y)^2 + f$$

with initial condition $y^0(x)$, then

$$\begin{aligned} \|u - y\|^2_{\frac{11}{6}^+}(t) &\leq [3\|u^0 - y^0\|^2_{\frac{11}{6}^+} + 3\|\sum_{k \neq 0} h_k^{1/2} A_t^k e_k - K * f\|^2_{\frac{11}{6}^+} \quad (5.3) \\ &+ \delta^2 C_1 \text{ess sup}_{s \in [t-\delta, t]} (\|u\|^2_{\frac{11}{6}^+} + \|y\|^2_{\frac{11}{6}^+})] e^{C_2 \int_0^{t-\delta} (1 + \|u\|^2_{\frac{11}{6}^+} + \|y\|^2_{\frac{11}{6}^+}) ds}, \end{aligned}$$

where C_1 and C_2 are constants and δ can be made arbitrarily small. The A_t^ks are the oscillatory Ornstein–Uhlenbeck-type processes (2.8) and K is the oscillatory kernal in (2.6).

Proof. We subtract the integral equation for y from that of u

$$\begin{aligned} u &= u^0 + \sum_{k\neq 0} h_k^{1/2} A_t^k e_k + K * (-u\cdot\nabla u + \nabla\Delta^{-1}\mathrm{tr}(\nabla u)^2) \\ y &= y^0 + K*f + K*(-y\cdot\nabla y + \nabla\Delta^{-1}\mathrm{tr}(\nabla y)^2) \end{aligned}$$

Thus

$$\begin{aligned} \|u-y\|^2_{\frac{11}{6}+}(t) &\leq [3\|u^0-y^0\|^2_{\frac{11}{6}+} + 3\|\sum_{k\neq 0} h_k^{1/2} A_t^k e_k - K*f\|^2_{\frac{11}{6}+} \\ &\quad +3\|K*(-w\nabla u - y\nabla w + \nabla\Delta^{-1}\mathrm{tr}\nabla\alpha\cdot\nabla w)\|^2_{\frac{11}{6}+}], \end{aligned}$$

where $w = u-y$ and $\alpha = u+y$. Now the same estimates as in Theorem 4.5 give

$$\begin{aligned} \|u-y\|^2_{\frac{11}{6}+}(t) &\leq 3\|u^0-y^0\|^2_{\frac{11}{6}+} + 3\|\sum_{k\neq 0} h_k^{1/2} A_t^k e_k - K*f\|^2_{\frac{11}{6}+} \\ &\quad +C_1\delta^2 \mathrm{ess\ sup}_{s\in[t-\delta,t]}(\|u\|^2_{\frac{11}{6}+} + \|y\|^2_{\frac{11}{6}+}) \\ &\quad +C_2\int_0^{t-\delta}(1+\|u\|^2_{\frac{11}{6}+} + \|y\|^2_{\frac{11}{6}+})(\|u-y\|^2_{\frac{11}{6}+})ds \end{aligned}$$

Then Grönwall's inequality gives (5.3). □

6. The existence of the invariant measure

In this section we will consider the stochastic Navier–Stokes equation

$$dw = (\nu\Delta w - w\cdot\nabla w + \nabla\Delta^{-1}\mathrm{tr}(\nabla w)^2)dt + \sum_{k\neq 0} h_k^{1/2} d\beta_t^k e_k \tag{6.1}$$

with initial data

$$w(x,0) = U_1 j_1 - A\sin(\Omega t+\theta)j_2 + A\cos(\Omega t+\theta)j_3 + u^0(x)$$

We will use that the solutions $u(x,t)$, where $w(x,t) = Uj_1 - A\sin(\Omega t+\theta)j_2 + A\cos(\Omega t+\theta)j_3 + u(x,t)$, exist in $L^2(\Omega,\mathcal{F},\mathbb{P};H^{\frac{11}{6}^+})$, by Theorem 5.2. $H^{\frac{11}{6}^+}(\mathbb{T}^3) = W^{(\frac{11}{6}^+,2)}$ is the Sobolev space based on L^2. By Theorem 5.2 the equation (6.1) defines a flow on the complete metric space

$$\begin{aligned} W = \{u\in L^2(\Omega,\mathcal{F},\mathbb{P};H^{\frac{11}{6}^+})|E(\|u\|^2_{\frac{11}{6}+}) \\ \leq (1-C(\frac{1}{B^2}+\delta^{\frac{1}{6}^-}))^{-1}[\sum_{k\neq 0}\frac{(1+(2\pi|k|)^{\frac{11}{3}^+})}{\pi^2\nu|k|^2}h_k + \frac{C'}{B}]\} \end{aligned}$$

This is the physical situation we are interested in, namely fully developed turbulence with nontrivial mean flow and rotation, see (2.3), and it applies to many if not most turbulent fluids, see [20, 21].

Since by Corollary 5.3, we can even take the initial data $u^0(x) \in \dot{L}^2(\mathbb{T}^3)$,[3] the integral equation

$$u(x,t) = K(t) * u^0(x) + u_o(x,t) - \int_0^t K(t-s) * (u \cdot \nabla u - \nabla \Delta^{-1} \mathrm{tr}(\nabla u)^2)\, ds,$$

with $u^0 \in \dot{L}^2(\mathbb{T}^3)$ and $u_o = \sum_{k \neq 0} h_k^{1/2} A_t^k e_k$, defines a map from a bounded set in $\dot{L}^2(\mathbb{T}^3)$ onto W. We define V to be the preimage of W in $\dot{L}^2(\mathbb{T}^3)$. V is also a complete metric space with the distance on V defined by the $\dot{L}^2(\mathbb{T}^3)$ norm.

More concretely, we can consider the initial value problem on V,

$$\begin{aligned} du = (\nu \Delta u \qquad & -U\partial_{x_1} u + A\sin(\Omega t + \theta)\partial_{x_2} u - A\cos(\Omega t + \theta)\partial_{x_3} u \\ & -u \cdot \nabla u + \nabla \Delta^{-1} \mathrm{tr}(\nabla u)^2) dt + \sum_{k \neq 0} h_k^{1/2} d\beta_t^k e_k \end{aligned} \tag{6.2}$$

$$u(x,0) \quad = \quad u^0(x) \in V \subset \dot{L}^2(\mathbb{T}^3)$$

Then by Theorem 5.2 and Corollary 5.3 the initial value problem (6.2) defines a flow on V.

If ϕ is a bounded function on V then the invariant measure $d\mu$ for the SPDE (2.3) is given by the limit

$$\lim_{t \to \infty} E(\phi(u(\omega,t))) = \int_V \phi(u) d\mu(u)$$

In this section we proof that this limit exists and is unique. We prove below that the limit exist in the $H^{\frac{11}{6}^+}(\mathbb{T}^3)$ norm on W but since it dominates the $\dot{L}^2(\mathbb{T}^3)$ norm on V the conclusions will follow for V.

Theorem 6.1. *The integral equation (5.2) possesses a unique invariant measure.*

Corollary 6.2. *The invariant measure $d\mu$ is ergodic and strongly mixing.*

The corollary follows immediately from Doob's Theorem on invariant measures, see for example [25].

We prove the theorem in three lemmas. First we define a transition probability

$$P_t(u^0, \Gamma) = \mathcal{L}(u(u^0,t))(\Gamma), \quad \Gamma \subset \mathcal{E},$$

where $\mathcal{L}$ is the law of $u(t)$, u^0 is the initial condition and $\mathcal{E}$ is the natural σ algebra of V. The action of P_t on the bounded function ϕ on V can be written as

$$P_t \phi = BM(\phi(u(u^0,t))) = \int_V \phi(u) \pi_t(u^0, du),$$

BM denoting the Brownian mean over the Brownian motions in equation (6.2) and π_t is the corresponding measure on V. Then

$$R_T(u^0, \cdot) = \frac{1}{T} \int_0^T P_t(u^0, \cdot) dt$$

[3]dot denotes mean zero

is a probability measure on V. By the Krylov–Bogoliubov theorem, see [25], if the sequence of measures R_T is tight then the invariant measure $d\mu$ is the weak limit

$$d\mu(\cdot) = \lim_{T\to\infty} \frac{1}{T}\int_0^T P_t(u^0,\cdot)dt$$

Namely,

$$R_T^* d\nu(\Gamma) = \int_V R_T(u^0,\Gamma)d\nu(u^0)$$

and

$$< R_T^*\nu, \phi >= \int_V \phi(u^0)R_T(u^0,\Gamma)d\nu(u^0) \to \int_V \phi(u^0)d\mu(u^0)$$

as $T \to \infty$.

Lemma 6.3. *The sequence of measures*

$$\frac{1}{T}\int_0^T P_t(u^0,\cdot)dt \tag{6.3}$$

is tight.

Proof. By the inequality (4.5)

$$\frac{1}{T}\int_0^T E(\|u\|^2_{\frac{11}{6}+})(t)dt \le C$$

The complete metric space W is relatively compact in V so it suffices to show that $u(t)$ lies in a bounded set in W almost surely, or for all $\epsilon > 0$ there exists an R such that,

$$\frac{1}{T}\int_0^T \mathbb{P}(\|u(t)\|^2_{\frac{11}{6}+} < R)dt > 1-\epsilon$$

for $T \ge 1$. But this follows from Chebychev's inequality, similarly as in Lemma 4.13, namely,

$$\frac{1}{T}\int_0^T \mathbb{P}(\|u(t)\|_{\frac{11}{6}+} \ge R)dt \le \frac{1}{R}C < \epsilon$$

for R sufficiently large. By Corollary 5.3 we can take the initial data in V. This proves that the sequence of measures (6.3) is tight. $\square$

Next we state a lemma proving the strong Feller property, see [25].

Lemma 6.4. *The Markovian semigroup P_t generated by the integral equation (5.2) on V is strongly Feller.*

The proof of the lemma is presented in [5].

Finally we prove irreducibility, see [25], of P_t. The proof of this lemma is an application of stochastic control theory.

Lemma 6.5. *The Markovian semigroup P_t generated by the integral equation (5.2) is irreducible.*

Proof. We first consider the linear deterministic equation

$$z_t + \mathbf{U}\cdot\nabla z = \nu\Delta z + w(x,t)$$
$$z(x,0)=0, \quad z(x,T)=b(x)$$

and the deterministic equation

$$\begin{aligned} y_t + \mathbf{U}\cdot\nabla y = \nu\Delta y - y\cdot\nabla y + \nabla\Delta^{-1}\mathrm{tr}(\nabla y)^2 + Qh(x,t) \\ y(x,0)=0, \quad y(x,T)=b(x)\,, \end{aligned} \tag{6.4}$$

where $Q : H^{-1} \to H^{\frac{11}{6}^+}$, both spaces have mean zero and kernel Q is empty. We will define the operator Q by a map from the coefficients (vectors) in an element $\sum_{k\neq 0} f_k e_k$ in $H^{-1}(\mathbb{T}^3)$ to the coefficients in the sum $\sum_{k\neq 0} h_k^{1/2} A_t^k e_k$, where the A_t^k are the oscillatory Ornstein–Uhlenbeck-type processes from (2.8). This map can be defined by an invertible matrix $h_k^{1/2} = Q_k f_k$, for example $Q_k = |k|^{-p} I_3$, where I_3 is the three by three identity matrix and p is a positive rational number, since all the coefficients in the latter sum satisfy $h_k^{1/2} \neq 0$. Then it is easy to check that kernel $Q = 0$.

We can pick a function $w \in C([0,T];W)$ such that $z(x,T)=b(x)$ and a corresponding function $h \in L^2([0,T];H^{-1}(\mathbb{T}^3))$. Namely, $Qh = z\cdot\nabla z - \nabla\Delta^{-1}\mathrm{tr}(\nabla z)^2 + w$, since the kernel of Q is empty; then $y = z$ is a solution of the deterministic Navier–Stokes equation (6.4) above. This means that (6.4) is exactly controllable on W, see Curtain and Zwartz [11].

Now we compare y and the solution u of the integral equation (5.1). By Lemma 5.4 we get that

$$\begin{aligned} \|u-y\|^2_{\frac{11}{6}^+}(t) \quad &\leq \quad [3\|u^0 - y^0\|^2_{\frac{11}{6}^+} + 3\|\sum_{k\neq 0} h_k^{1/2} A_t^k e_k - Qh\|^2_{\frac{11}{6}^+} + \\ &\delta^2 C_1 \quad \text{ess sup}_{s\in[t-\delta,t]}(\|u\|^2_{\frac{11}{6}^+} + \|y\|^2_{\frac{11}{6}^+})]e^{C_2\int_0^{t-\delta}(1+\|u\|^2_{\frac{11}{6}^+}+\|y\|^2_{\frac{11}{6}^+})ds} \end{aligned}$$

By Lemma 4.13, for $\gamma > 0$

$$\mathbb{P}(\|u\|^2_{\frac{11}{6}^+} + \|y\|^2_{\frac{11}{6}^+}) \leq R) > 1 - \frac{\gamma}{2}$$

if

$$E(\|u\|^2_{\frac{11}{6}^+} + \|y\|^2_{\frac{11}{6}^+})/R \leq \frac{\gamma}{2}$$

Then

$$\begin{aligned} E(\|u-y\|^2_{\frac{11}{6}^+}(T)) \;&\leq\; [3E(\|\sum_{k\neq 0} h_k^{1/2} A_T^k e_k - Qh\|^2_{\frac{11}{6}^+}) + \delta^2 C_1 R]e^{C_2(1+R)(T-\delta)} \\ &\leq\; \frac{\epsilon\gamma}{4} \end{aligned} \tag{6.5}$$

if δ is small enough, since $\sum_{k\neq 0} h_k^{1/2} A_t^k e_k$ is an oscillatory Ornstein–Uhlenbeck-type process with a non-degenerate covariance, whose (Gaussian) measure is full in $L^2([0,T]; H^{\frac{11}{6}^+})$. This implies that the probability

$$\begin{aligned}
\mathbb{P}(\|u(T)-b\|_{\frac{11}{6}^+} \le \epsilon \quad &\text{and} \quad \|u\|^2_{\frac{11}{6}^+} + \|y\|^2_{\frac{11}{6}^+}) \le R) \ge \\
\mathbb{P}\Big(\|u(T)-y(T)\|_{\frac{11}{6}^+} \le \frac{\epsilon}{2} \quad &\text{and} \quad \|y(T)-b(T)\|_{\frac{11}{6}^+} \le \frac{\epsilon}{2} \\
&\text{and} \quad \|u\|^2_{\frac{11}{6}^+} + \|y\|^2_{\frac{11}{6}^+} \le R\Big) \\
&\ge \quad 1 - \frac{\gamma}{2} - \frac{\gamma}{2} = 1-\gamma > 0
\end{aligned}$$

by (6.5) and Chebychev's inequality, since (6.4) is exactly controllable. It also implies that

$$\mathbb{P}(|u(T)-b|_2^2) \le \epsilon) > 0$$

□

Proof of Theorem 6.1 and Corollary 6.2

Proof. Theorem 6.1 and Corollary 6.2 are now easily proven in the following manner. If the Markovian semigroup P_t is strongly Feller and invariant, as it is by Lemmas 6.4 and 6.5, it is also t-regular. This means that the probability measures $P(u_o(s),\cdot)$ are all equivalent for $s \ge t$, and then by Doob's Theorem for invariant measures, see [25], the invariant measure is unique and strongly mixing. □

6.1. **Kolmogorov's scaling.** In 1941, Kolmogorov [14] formulated his famous scaling theory of the inertial range in turbulence, stating that the second-order structure function, scales as

$$S_2(x) = \langle |u(y+x)-u(y)|^2\rangle \sim (\epsilon|x|)^{2/3},$$

where $y, y+x$ are points in a turbulent flow field, u is the component of the velocity in the direction of x, ϵ is the mean rate of energy dissipation, and the angle brackets denote an (ensamble) average. A Fourier transform yields the Kolmogorov–Obukhov power spectrum in the inertial range

$$E(k) = C\epsilon^{2/3}k^{-5/3},$$

where C is a constant, k is the wave number and $E(k)$ denotes the energy density in Fourier space. These results form the basis of turbulence theory. The following theorem proves the basic statement in Kolmogorov's statistical theory of turbulence.

Theorem 6.6. *The second structure function of turbulence satisfies the estimate*

$$\begin{aligned}
S_2(x,t) &= E[|u(x+y,t)-u(y,t)|_2^2] \\
&= \int_V |u(x+y,t)-u(y,t)|_2^2 d\mu(u) \le \ C|x\cdot(\mathbf{L}-x)|^{2/3}, \qquad (6.6)
\end{aligned}$$

where C is a constant and $\mathbf{L}$ is a three vector giving the dimensions of the torus $\mathbb{T}^3$.

Proof. The proof is basically an amplification of Corollary 4.6. We write the difference as a Fourier series

$$u(x+y,t)-u(y,t)=\sum_{k\neq 0}\hat{u}(k)e^{2\pi ik\cdot y}(e^{2\pi ik\cdot x}-1)$$

By the Cauchy-Schwartz inequality

$$\begin{aligned}|u(x+y,t)-u(y,t)| &\leq (\sum_{k\neq 0}|k|^{3+2\gamma}|\hat{u}(k)|^2)^{1/2}(\sum_{k\neq 0}|k|^{-3-2\gamma}|e^{2\pi ik\cdot x}-1|^2)^{1/2}\\ &\leq \|u\|_{\frac{3}{2}+\gamma}(\sum_{k\neq 0}|k|^{-3-2\gamma}|e^{2\pi ik\cdot x}-1|^2)^{1/2}\end{aligned}$$

We use the integral test to estimate the last series

$$\begin{aligned}(\sum_{k\neq 0}|k|^{-3-2\gamma}|e^{2\pi ik\cdot x}-1|^2) &\leq C\int_{\mathbb{R}^3}|k|^{-3-2\gamma}|e^{2\pi ik\cdot x}-1|^2dk\\ &= C(4\pi^2\int_{|k|\leq\frac{1}{|x|}}|x|^2|k|^2|k|^{-3-2\gamma}dk+4\int_{|k|\geq\frac{1}{|x|}}|k|^{-3-2\gamma}dk)\\ &= C(4\pi^2\frac{|x|^{2\gamma}}{2-2\gamma}+4\frac{|x|^{2\gamma}}{2\gamma})\end{aligned}$$

for $x_j\leq L_j/2,\ j=1,2,3$. Now squaring and taking the expectation we get that

$$E[|u(x+y,t)-u(y,t)|^2]\leq CE[\|u\|^2_{\frac{3}{2}+\gamma}]\ |x|^{2\gamma}$$

for $x_j\leq L_j/2,\ j=1,2,3$. Moreover by making the same estimate for the variable $z=\mathbf{L}-x$, where the three-vector $\mathbf{L}$ has the entries $L_j,\ j=1,2,3$, we get the estimate

$$E[|u(z+y,t)-u(y,t)|^2]\leq CE[\|u\|^2_{\frac{3}{2}+\gamma}]\ |\mathbf{L}-x|^{2\gamma}$$

for $x_j\geq L_j/2,\ j=1,2,3$. Combining the two estimates we obtain the estimate

$$E[|u(x+y,t)-u(y,t)|^2]\leq CE[\|u\|^2_{\frac{3}{2}+\gamma}]\ |x\cdot(\mathbf{L}-x)|^{2\gamma}$$

and then choosing $\gamma=\frac{1}{3}^+$ and applying the estimated (4.5) to $E[\|u\|^2_{\frac{11}{6}+}]$ we get the estimate (6.6). □

Remark 6.7. The estimate (6.6) is not sharp due to intermittency, as pointed out by Landau and discussed by Kolmogorov [15].

Theorem 6.8. *There exist solutions of the stochastic Navier–Stokes equation (2.3) with an expectation of the $H^{\frac{11}{6}^+}$ norm that is uniformly bounded for every $t\in\mathbb{R}^+$, but whose expectation of the H^{2^-} norm is infinite for every $t\in\mathbb{R}^+$.*

Proof. Suppose that the expectation of the $H^{\frac{11}{6}+}$ norm of u is finite by Theorem 4.5. Then a similar argument as lead to inequality (4.8) gives the inequality

$$\begin{aligned}\|u\|^2_{\frac{11}{6}+\sigma} &\geq 9\sum_{k\neq 0}(1+(2\pi|k|)^{\frac{11}{3}+2\sigma})h_k|A_t^k|^2\\ &\quad -C(\epsilon+\delta^{\frac{1}{6}-\sigma})\text{ess sup}_{s\in[0,t]}\|u\|^2_{\frac{11}{6}+}\end{aligned}$$

Now if

$$E[\sum_{k\neq 0}(1+(2\pi|k|)^{\frac{11}{3}+2\sigma})h_k|A_t^k|^2]=\sum_{k\neq 0}\frac{(1+(2\pi|k|)^{\frac{11}{3}+2\sigma})}{4\pi^2|k|^2}h_k=\infty$$

for $0<\sigma<\frac{1}{6}$ then it follows that

$$E[\|u\|^2_{\frac{11}{6}+\sigma}]=\infty$$

also. □

7. Hopf's equation for the invariant measure

In 1952 Hopf [12] found an equation for the characteristic function of the invariant measure, whose existence was proven in section 6. His equation can be written as a functional differential equation for invariant measure and we will write it in a slightly general form

$$w_t=\frac{1}{2}\mathrm{tr}[C\Delta_u w]+\langle Au,\nabla_u w\rangle\,, \tag{7.1}$$

where C is a matrix of coefficients, whose trace consists of the small noise coefficients in (2.4) squared and A is the linearized Navier–Stokes operator,

$$Af=\nu\Delta f-u\cdot\nabla f-f\cdot\nabla u+2\nabla\Delta^{-1}\mathrm{tr}(\nabla u\cdot f)$$

w lives on the space of initial data $u\in H^1(\mathbb{T}^1)$ and the functional derivative ∇_u is with respect to functions in this space.

The equation (7.1) is linear and it is solved and the solution analyzed in [7]. This involves a very complicated limit as $t\to\infty$ and we will not repeat this analysis here.

Once the solution of (7.1) is found one can write down a formula for the large noise (1.3) and thus close the loop for the mathematical proof of Kolmogorov's statistical theory. Of course this means that the intermittency corrections discussed in the previous section show up in the decay of the coefficients $h_k^{1/2}$.

7.1. **An approximation of the measure.** We give an idea of the form and properties of the invariant measure by the following approximation of it, see [4]. Consider the following approximation of the "eddy viscosity". Let P denote the projection operator, then we will model the difference between the projection of the inertial terms and the intertial terms themselves as

$$P[u\cdot\nabla u]-u\cdot\nabla u\approx\sum_{k\neq 0}g_k^{1/2}d\beta_t^k e_k(x)\cdot\nabla u$$

This expression is of course not exact, but the modeling is motivated by numerical simulations, where an analogous difference the "eddy viscosity", is shown to depend on the gradient ∇u.

Now the initial value problem for the Navier–Stokes equation can be written in the form

$$\frac{\partial u}{\partial t}+w\cdot\nabla u=\nu\Delta u+\sum_{k\neq 0}h_k^{1/2}d\beta_t^k e_k(x) \tag{7.2}$$

$$u(x,0)=f(x)\,,$$

where

$$w(x,t) = u + \sum_{k\neq 0} g_k^{1/2} d\beta_t^k e_k(x)$$

and we use the same notation for the divergence free $k \cdot h_k^{1/2} = 0$ vectors as for the original $h_k^{1/2}$. Then introducing the Ito diffusion

$$dX_t = -w(X_t, t)dt + \sqrt{2\nu} dB_t$$

we can write the solution of (7.2) of the form

$$u(x,t) = E[f(X_t)] + \sum_{k\neq 0} h_k^{1/2} \int_0^t E[e_k(X_{t-s})] d\beta_s^k$$

Now by Girsanov's theorem, see [22], we can rewrite u in the form

$$u(x,t) = E[f(B_t)M_t] + \sum_{k\neq 0} h_k^{1/2} \int_0^t E[e_k(B_{t-s})M_{t-s}] d\beta_s^k ,$$

where

$$M_t = exp\{-\int_0^t w(B_s, s) \cdot dB_s - \frac{1}{2} \int_0^t |w(B_s, s)|^2 ds\}$$

This implies that (7.2) has the invariant measure

$$d\mu = \lim_{t\to\infty} M_t d \left[\mathcal{N}(e^{\nu\Delta t}, \sqrt{2\nu}) * \mathcal{N}(B_t^\infty, Q_t)\right], \tag{7.3}$$

where $e^{\nu\Delta t}$ is the heat semigroup, B_t^∞ is the evolution operator of the infinite-dimensional Brownian motion and the variance Q_∞ is

$$Q_\infty^{-1} = \sum_{k\neq 0} \frac{h_k}{2\nu\lambda_k}$$

the coefficients being $h_k = |h_k^{1/2}|^2$. The statistical theory of (7.2) is determined by the invariant measure (7.3).

We can also write the approximate invariant measure in terms of densities

$$d\mu \approx e^{\{-\int_0^t w(x,s)\cdot dx - \frac{1}{2}\int_0^t |w(x,s)|^2 ds\}} \frac{e^{-\frac{|x|^2}{2\nu}}}{\sqrt{2\nu}} dx \prod_{k\neq 0} \frac{e^{-\frac{h_k \hat{u}_k^2}{2\nu\lambda_k}}}{\sqrt{2\nu\lambda_k/h_k}} d\hat{u}_k$$

for t large, where $\hat{u}_k$ are the Fourier coefficients of u.

In numerical simulation and fluid experiments the approximate velocity w will have similar statistical properties as the real velocity u. Thus w can be approximated by simulated or measured values of the fluid velocity u itself. This gives and implicit formula for the approximation of the invariant measure,

$$d\mu \approx e^{\{-\int_0^t u(x,s)\cdot dx - \frac{1}{2}\int_0^t |u(x,s)|^2 ds\}} \frac{e^{-\frac{|x|^2}{2\nu}}}{\sqrt{2\nu}} dx \prod_{k\neq 0} \frac{e^{-\frac{h_k \hat{u}_k^2}{2\nu\lambda_k}}}{\sqrt{2\nu\lambda_k/h_k}} d\hat{u}_k$$

for t large. The decay of the coefficients $h_k^{1/2}$ that gives the Kolmogorov scaling, or the Sobolev space $H^{\frac{11}{6}^+}$ corresponds to the infinite product on the right hand side. The intermittency correction come from the interplay of the Martingale prefactor $M_t = e^{\{-\int_0^t u(x,s)\cdot dx - \frac{1}{2}\int_0^t |u(x,s)|^2 ds\}}$ and the Gaussian density $\frac{e^{-\frac{|x|^2}{2\nu}}}{\sqrt{2\nu}}$.

Acknowledgement. This research was supported by a grant from the UCSB Academic Senate that is gratefully acknowledged.

References

1. A. Babin, A. Mahalov, and B. Nicolaenko, *Long-time averaged Euler and Navier-Stokes equations for rotation fluids*, In Structure and Dynamics of non-linear waves in Fluids, 1994 IUTAM Conference, K. Kirehgässner and A. Mielke (eds), World Scientific (1995), 145–157.
2. ______, *Global splitting, integrability and regularity of 3d Euler and Navier-Stokes equation for uniformely rotation fluids*, Eur. J. Mech. B/Fluids **15** (1996), no. 2, 08312.
3. B. Birnir, *Turbulence of a unidirectional flow*, Proceedings of the Conference on Probability, Geometry and Integrable Systems, MSRI Dec. 2005, MSRI Publications, Cambridge Univ. Press **55** (2007).
4. ______, *Approximations of Turbulent Invariant Measures*, Proceedings of the iTi Turbulence Meeting, Bertinoro Italy, eds. Joachim Peinke and Martin Overlack (2008).
5. ______, *The existence and uniqueness of stochastic solutions of the stochastic Navier-Stokes equation*, CNLS preprint (2008).
6. ______, *Turbulent rivers*, Quart. Appl. Math. **66** (2008), no. 3, 565–594.
7. ______, *The Invariant Measure of Turbulence*, CNLS preprint (2009).
8. B. Birnir, K. Mertens, V. Putkaradze, and P. Vorobieff, *Meandering fluid streams in the presence of flow-rate fluctuations*, Phys. Rev. Letters **101** (2008), 114501, DOI: 10.1103/PhysRevLett.101.114501.
9. ______, *Morphology of a stream flowing down an inclined plane. part 2. meandering,*, J. Fluid Mech. **607** (2008), 401 − 417.
10. V.T. Chow, *Open channel hydraulics*, McGraw-Hill, New York, 1959.
11. R.F. Curtain and H. Zwart, *An introduction to infinite-dimensional linear systems theory*, Springer, New York, 1995.
12. E. Hopf, *Statistical hydromechanics and functional calculus*, Rat. Mech. Anal. **1** (1952), 87–123.
13. S. Hou, B. Birnir, and N. Wellander, *Derivation of the viscous Moore-Greitzer equation for aeroengine flow*, Journ. Math. Phys. **48** (2007), 065209.
14. A.N. Kolmogorov, *The local structure of turbulence in incompressible viscous fluid for very large reynolds number*, Dokl. Akad. Nauk SSSR **30** (1941), 9–13.
15. ______, *Refined description on the local structure of turbulence in an incompressible viscous fluid at high reynolds numbers*, Mécanique du la turbulence, Coll. Intern. du CNRS du Marseille, Ed CRNS, Paris (1962), 447–458.
16. S. Kuksin and A. Shirikyan, *A coupling approach to randomly forced nonlinear pdes*, Comm. Math. Phys. **221** (2001), 351–366.
17. J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. **63** (1934), no. 3, 193–248.
18. H.P. McKean, *Turbulence without pressure: Existence of the invariant measure*, Methods and Applications of Analysis **9** (2002), no. 3, 463–468.
19. C. Meneveau and J. Katz, *Scale invariance and turbulence models for large-eddy simulations*, Annu. Rev. Fluid Mech. **32** (2000), 1–32.
20. A.S. Momin and A.M. Yaglom, *Statistical fluid mechanics*, vol. 1, MIT Press, Cambridge, MA, 1971.
21. ______, *Statistical fluid mechanics*, vol. 2, MIT Press, Cambridge, MA, 1975.
22. B. Oksendal, *Stochastic differential equations*, Springer, New York, 1998.

23. L. Onsager, *The distribution of energy in turbulence*, Phys. Rev. **68** (1945), 285.
24. ———, *Statistical hydrodynamics*, Nuovo Cimento. **6** (1945), no. 2, 279–287.
25. G. Da Prato and J. Zabczyk, *Ergodicity for infinite dimensional systems*, Cambridge University Press, Cambridge UK, 1996.
26. Y. Sinai, *Burgers equation driven by a periodic stochastic flow*, Ito's Stochastic Calculus and Probability Theory, Springer New York (1996), 347–353.
27. J. Smagorinsky, *General circulation experiments with primitive equations*, Mon. Weather Rev. **91** (1963), 99–165.

[1] Center for Complex and Nonlinear Science, and, Department of Mathematics, University of California, Santa Barbara, CA 93106, USA., and, the University of Iceland, Reykjavík

E-mail address: birnir@math.ucsb.edu