The Kolmogorov-Obukhov Theory of Turbulence

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Outline

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The Deterministic Navier-Stokes Equations

- A general incompressible fluid flow satisfies the Navier-Stokes Equation

$$\begin{aligned} u_t + u \cdot \nabla u &= \nu \Delta u - \nabla p \\ u(x,0) &= u_0(x) \end{aligned}$$

with the incompressibility condition

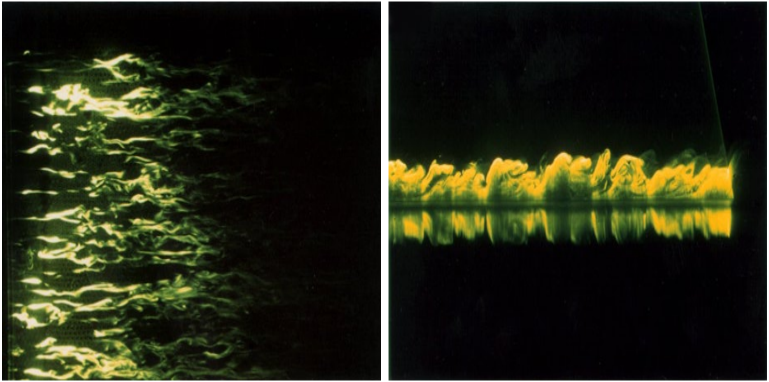
$$\nabla \cdot u = 0,$$

- Eliminating the pressure using the incompressibility condition gives

$$\begin{aligned} u_t + u \cdot \nabla u &= \nu \Delta u + \nabla \Delta^{-1} \mathrm{trace}(\nabla u)^2 \\ u(x,0) &= u_0(x) \end{aligned}$$

- The turbulence is quantified by the dimensionless Reynolds number $R = \frac{UL}{\nu}$

Boundary Layers and Turbulence



The Stochastic Navier-Stokes Equations

- In turbulent fluids the laminar solution is unstable
- Small noise is magnified by the fluid instability and the saturated by the nonlinearities in the flow and in the Navier-Stokes equations
- It was pointed out by Kolmogorov [8] that it is more useful in turbulent flow to consider the velocity $u(x,t)$ to be a stochastic process
- Then it satisfies the stochastic Navier-Stokes equation

$$\begin{aligned} du = (-u \cdot \nabla u &+ \nu \Delta u + \nabla \Delta^{-1} \mathrm{trace}(\nabla u)^2) dt + df_t \\ u(x,0) &= u_0(x) \end{aligned}$$

- df_t is the noise in fully developed turbulence

Outline

The central limit theorem

- We construct the noise using the central limit theorem
- Split the torus $\mathbb{T}^3$ into little boxes and consider the dissipation to be a stochastic process in each box
- By the central limit theorem the average

$$S_n = \frac{1}{n} \sum_{j=1}^{n} p_j$$

  converges to a Gaussian random variable as $n \to \infty$
- This holds for any Fourier component (e_k) and the result is the infinite dimensional Brownian motion

$$df_t^1 = \sum_{k \neq 0} c_k^{\frac{1}{2}} db_t^k e_k(x)$$

Intermittency and the large deviation principle

- In addition we get intermittency of the dissipation
- If these excursions are completely random then they are modeled by Poisson process with the rate λ
- Applying the large deviation principle, we get exponentially distributed processes, with rate $|k|^{1/3}$
- This also holds in the direction of each Fourier component and gives the noise

$$df_t^2 = \sum_{k \neq 0} d_k d\nu_t^k e_k(x)$$

Intermittency and velocity fluctuation

- So far our noise is additive. There is also multiplicative noise due to velocity fluctuation
- The multiplicative noise, models the excursion (jumps) in the velocity gradient
- If these jumps are completely random they should be modeled by a Poisson process η_t^k
- N_t^k denotes the integer number of velocity excursion, associated with kth wavenumber, that have occurred at time t.
- The differential $dN^k(t) = N^k(t+dt) - N^k(t)$ denotes these excursions in the time interval $(t, t+dt]$.
- The process

$$df_t^3 = \sum_{k \neq 0}^{M} \int_{\mathbb{R}} h_k(t,z) \bar{N}^k(dt,dz),$$

gives the multiplicative noise term

Stochastic Navier-Stokes with Turbulent Noise

- Adding the two types of additive noise and the multiplicative noise we get the stochastic Navier-Stokes equations describing fully developed turbulence

$$
\begin{aligned}
du &= (\nu\Delta u - u\cdot\nabla u + \nabla\Delta^{-1}\mathrm{tr}(\nabla u)^2)dt \\
&+ \sum_{k\neq 0} c_k^{\frac{1}{2}} db_t^k e_k(x) + \sum_{k\neq 0}^{M} d_k d\nu_t^k e_k(x) \\
&+ u(\sum_{k\neq 0}^{M} \int_{\mathbb{R}} h_k \bar{N}^k(dt,dz)) \\
u(x,0) &= u_0(x)
\end{aligned}
$$

- Each Fourier component e_k comes with its own Brownian motion b_t^k and Poisson process ν_t^k

Computation of the measure

Now the linearized equation

$$\begin{aligned} dz &= (\nu\Delta z - u\cdot\nabla z - z\cdot\nabla u + 2\nabla\Delta^{-1}\mathrm{tr}(\nabla u \nabla z)dt \\ &+ \sum_{k\neq 0}(c_k^{\frac{1}{2}} db_t^k + d_k d\nu_t^k)e_k(x) \qquad (1) \\ &+ z(\sum_{k\neq 0}\int_{\mathbb{R}} h_k(t,z)\bar{N}^k(dt,dz)) \\ z(0) &= z_0 \end{aligned}$$

has (almost) the same invariant measure
as the stochastic Navier-Stokes equation for velocity differences.

Solution of the Stochastic Linearized Navier-Stokes

- We solve (1) using the additional help of the Feynmann-Kac formula, and Cameron-Martin (or Girsanov's Theorem)
- The solution is

$$
\begin{aligned}
z &= e^{Kt} e^{\int_0^t dq} M_t z^0 \\
&+ \sum_{k \neq 0} \int_0^t e^{K(t-s)} e^{\int_s^t dq} M_{t-s} (c_k^{1/2} d\beta_s^k + d_k d\nu_s^k) e_k(x)
\end{aligned}
$$

- K is the operator $K = \nu\Delta + 2\nabla\Delta^{-1} tr(\nabla u \nabla)$
- K generates a semi-group by the perturbation theory of linear operators (Kato)

Cameron-Martin and Feynmann-Kac

- M_t is the Martingale

$$M_t = exp\{-\int_0^t u(B_s, s) \cdot dB_s - \frac{1}{2}\int_0^t |u(B_s, s)|^2 ds\}$$

- Using M_t as an integrating factor eliminates the inertial terms from the equation (1)
- The Feynmann-Kac formula gives the exponential of a sum of terms of the form

$$\int_s^t dq^k = \int_0^t \int_{\mathbb{R}} \ln(1 + h_k) N^k(dt, dz) - \int_0^t \int_{\mathbb{R}} h_k m^k(dt, dz),$$

by a computation similar to the one that produces the geometric Lévy process, see [12], m^k the Lévy measure.

The log-Poisson processes

- The form of the processes $e^{\int_0^t \int_{\mathbb{R}} \ln(1+h_k) N^k(dt,dz) - \int_0^t \int_{\mathbb{R}} h_k m^k(dt,dz)}$ was found by She and Leveque [13], for $h_k = \beta - 1$,
- N_t^k counts the number of jumps, with the mean

$$\int_{\mathbb{R}} m^k(t,dz) = -\frac{\gamma \ln|k|}{\beta - 1}, \quad \int_0^t \int_{\mathbb{R}} h_k N^k(ds,dz) = N_t^k \ln(\beta)$$

- It was pointed out by She and Waymire [14] and by Dubrulle [6] that they are log-Poisson processes.

$$e^{\int_0^t \int_{\mathbb{R}} \ln(1+h_k) N^k(dt,dz) - \int_0^t \int_{\mathbb{R}} h_k m^k(dt,dz)} = e^{N_t^k \ln \beta + \gamma \ln|k|} = |k|^{\gamma} \beta^{N_t^k}$$

The Spectral Theory of the Operator K

Suppose that

$$E(\|u\|^2_{\frac{3}{2}^+}) \leq C_1 \tag{2}$$

then the operator K generates contration semi-groups denoted e^{Kt}. We get using the bound [1],[4],

$$E(\|u\|^2_{\frac{11}{6}^+}(t)) \leq C \tag{3}$$

Lemma (The Inertial Range)

The spectrum of the operators K satisfies the estimate

$$|\lambda_k + \nu 4\pi^2 |k|^2| \leq C|k|^{2/3} \tag{4}$$

Computation of the structure functions

Lemma (The Kolmogorov-Obukov scaling)

The scaling of the structure functions is

$$S_p \sim C_p|x-y|^{\zeta_p},$$

where

$$\zeta_p = \frac{p}{3} + \tau_p = \frac{p}{9} + 2(1-(2/3)^{p/3})$$

$\frac{p}{3}$ *being the Kolmogorov scaling and* τ_p *the intermittency corrections. The scaling of the structure functions is consistent with Kolmogorov's* $4/5$ *law,*

$$S_3 = -\frac{4}{5}\varepsilon|x-y|$$

to leading order, were $\varepsilon = \frac{d\mathcal{E}}{dt}$ *is the energy dissipation*

The first few structure functions

$$S_1(x,y,t) = \frac{2}{C} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} d_k \frac{(1-e^{-\lambda_k t})}{|k|^{\zeta_1}} \sin(\pi k \cdot (x-y)).$$

$\sum_{k \in \mathbb{Z}^3 \setminus \{0\}} d_k < \infty$, and for $|x-y|$ small,

$$S_1(x,y,\infty) \sim \frac{2}{C} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} d_k |x-y|^{\zeta_1},$$

where $\zeta_1 = 1/3 + \tau_1 \approx 0.37$. Similarly

$$S_2(x,y,\infty) \sim \frac{2\pi^{\zeta_2}}{C} \sum_{k \in \mathbb{Z}^3} [c_k + \frac{2d_k^2}{C}]|x-y|^{\zeta_2},$$

when $|x-y|$ is small, where $\zeta_2 = 2/3 + \tau_2 \approx 0.696$.

The higher order structure functions

All the structure functions are computed in a similar manner. If $p = 2n+1$ is odd,

$$S_p = \frac{2^p}{C^p} \sum_{k \in \mathbb{Z}^3} {d_k}^p \frac{(1-e^{-2\lambda_k t})^p}{|k|^{\zeta_p}} \sin^n(\pi k \cdot (x-y))$$

to leading order in the lag variable $|x-y|$. If $p = 2n$ is even, S_p is

$$\sum_{k \in \mathbb{Z}^3} [\frac{2^n}{C^n} c_k^n \frac{(1-e^{-2\lambda_k t})^n}{|k|^{\zeta_p}} + \frac{2^p}{C^p} {d_k}^p \frac{(1-e^{-\lambda_k t})^p}{|k|^{\zeta_p}}] \sin^p(\pi k \cdot (x-y)),$$

to leading order in $|x-y|$.

Outline

The Kolmogorov-Obukov scaling hypothesis

- The Kolmogorov-Obukov scaling *with the intermittency corrections* τ_p, is

$$S_n(l) = C_p l^{\zeta_p}, \quad \zeta_p = \frac{p}{3} + \tau_p = \frac{p}{9} + 2(1-(2/3)^{p/3}) \quad (5)$$

  where l is the lag variable $l = |x-y|$.
- The coefficients C_p are not universal but depend on the c_ks and d_ks that in turn depend on the large eddies in the turbulent flow
- $C_p = \frac{2^p\pi^{\zeta_p}}{C^p}\sum_{k\in\mathbb{Z}^3\setminus\{0\}} {d_k}^p$ or $C_p = \frac{2^n\pi^{\zeta_p}}{C^n}\sum_{k\in\mathbb{Z}^3}[c_k^n + \frac{2^n}{C^n}{d_k}^p]$

Kolmogorov's refined scaling hypothesis

- In [9, 11] Kolmogorov and Obukhov presented their refined similarity hypothesis

$$S_p = C'_p < \tilde{\varepsilon}^p > l^{p/3}$$

where l is the lag variable and $\tilde{\varepsilon}$ is an averaged energy dissipation rate

- It can be shown, see [5], that by defining $\tilde{\varepsilon}$ appropriately, this gives

$$S_p = C'_p < \tilde{\varepsilon}^p > l^{p/3} = C_p l^{\zeta_p}$$

where the coefficients C'_p now *are universal*

-

$$S_p(t,T,l) = C_p l^{\zeta_p} + D_p(t) T^{\gamma_p}, \gamma_p = \frac{p}{6} + 3(1-(2/3)^{p/3})$$

The Invariant Measure and the Probability Density Functions (PDF)

- The above computation is a computation of a *test invariant measure*, that the real invariant measure should be absolutely continuous with respect to
- Hopf [7] write down a functional differential equation for the characteristic function of the invariant measure
- The quantity that can be compared directly to experiments is the PDF

$$E(\delta_j u) = E([u(x+s,\cdot) - u(x,\cdot)] \cdot r) = \int_{\infty}^{\infty} f_j(x) dx,$$

$j = 1$, if $r = \hat{s}$ is the longitudinal direction, and $j = 2$, $r = \hat{t}$, $t \perp s$ is a transversal direction

- Using Jacobi's identity and the asymptotics of the moments, we compute the PDF directly

Outline

Computing the PDF from the characteristic function

- Taking the characteristic functions of the measure of the stochastic processes in Equation (1), we get

$$\hat{f}(k) \sim k^{1-\zeta_1} e^{-\delta k}$$

- Translating this function and taking the inverse Fourier transform gives

$$f(x) \sim \frac{e^{-d|x|} e^{-bx}}{(x - i\delta)^{2-\zeta_1}}$$

Inserting a Gaussian

- The probability density function (PDF) of the components of the velocity increments is a generalized hyperbolic distribution, see Barndorff-Nilsen [2]
- Letting $\alpha, \delta \to \infty$, in the formulas for $f_j(x)$ below, in such a way that $\delta/\alpha \to \sigma$, we get that

$$f_j \to \frac{e^{-\frac{(x-\mu)^2}{2\sigma}}}{\sqrt{2\pi\sigma}} e^{\beta(x-\mu)}.$$

- The exponential tails of the PDF are caused by occasional sharp velocity gradients (rounded of shocks)
- The cusp at the origin is caused by the random and gentile fluid motion in the center of the ramps leading up to the sharp velocity gradients, see Kraichnan [10]

The Probability Density Function (PDF)

Lemma

The PDF is a generalized hyperbolic distribution, $\lambda = 1 - \zeta_1$:

$$f(x_j) = \frac{(\delta/\gamma)^{1-\zeta_1}}{\sqrt{2\pi}K_{1-\zeta_1}(\delta\gamma)} \frac{K_{1-\zeta_1}(\alpha\sqrt{\delta^2 + (x_j - \mu)^2})e^{\beta(x-\mu)}}{(\sqrt{\delta^2 + (x_j - \mu)^2}/\alpha)^{1-\zeta_1}} \qquad (6)$$

where $K_{1-\zeta_1}$ is modified Bessel's function of the second kind, $\gamma = \sqrt{\alpha^2 - \beta^2}$, ζ_1 the scaling exponent of S_1,

$$f(x) \sim \frac{(\delta/\gamma)^{1-\zeta_1}}{2\pi K_{1-\zeta_1}(\delta\gamma)} \frac{\Gamma(1-\zeta_1)2^{1-\zeta_1}e^{\beta\mu}}{(\delta^2 + (x-\mu)^2)^{1-\zeta_1}} \quad \text{for } x << 1$$

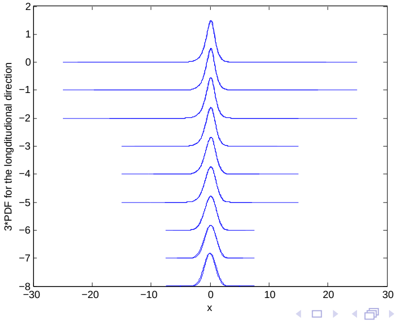
$$f(x) \sim \frac{(\delta/\gamma)^{1-\zeta_1}}{2\pi K_{1-\zeta_1}(\delta\gamma)} \frac{e^{\beta(x-\mu)}e^{-\alpha x}}{x^{3/2-\zeta_1}} \quad \text{for } x >> 1$$

Outline

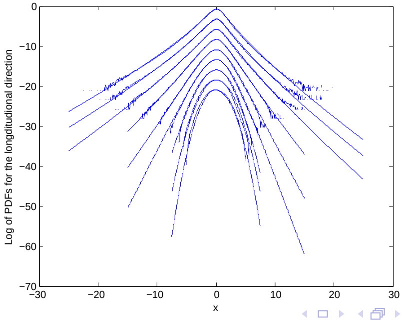
Existence and Uniqueness of the Invariant Measure

- We now compare the above PDFs with the PDFs found in simulations and experiments.
- The direct Navier-Stokes (DNS) simulations were provided by Michael Wilczek from his Ph.D. thesis, see [15].
- The experimental results are from Eberhard Bodenschatz experimental group in Göttingen.
- We thank both for the permission to use these results to compare with the theoretically computed PDFs.
- A special case of the hyperbolic distribution, the NIG distribution, was used by Barndorff-Nilsen, Blaesild and Schmiegel [3] to obtain fits to the PDFs for three different experimental data sets.

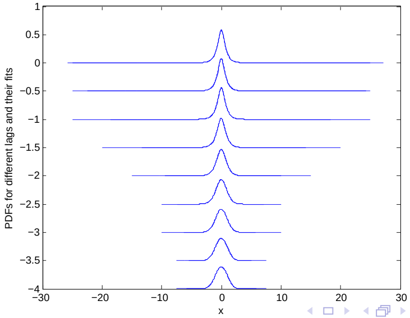
The PDF from simulations and fits for the longitudinal direction



The log of the PDF from simulations and fits for the longitudinal direction
Compare Fig. 4.5 in [15]

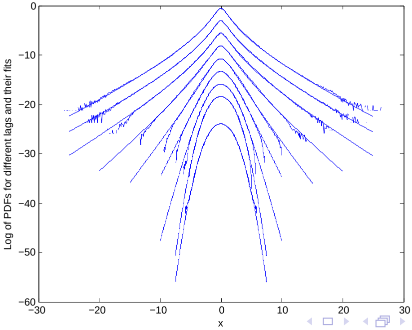


The PDF from simulations and fits for a transversal direction

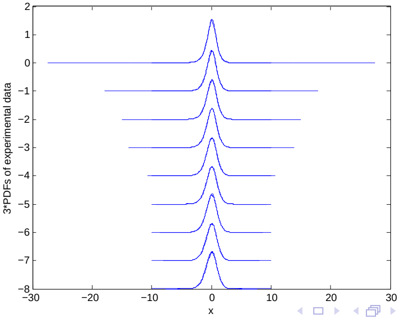


The log of the PDF from simulations and fits for the a transversal direction

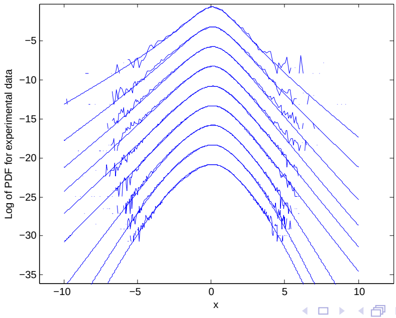
Compare Fig. 4.6 in [15]



The PDF from experiments and fits



The log of the PDF from experiments and fits



The Artist by the Water's Edge

Leonardo da Vinci Observing Turbulence

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