The Navier-Stokes Millennium Problem: Laminar versus Turbulent Flow

Björn Birnir

Center for Complex and Nonlinear Science
and
Department of Mathematics, UC Santa Barbara

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Outline

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The Deterministic Navier-Stokes Equations

- A general incompressible fluid flow satisfies the Navier-Stokes Equation

$$
\begin{aligned}
u_t + u \cdot \nabla u &= \nu \Delta u - \nabla p \\
u(x,0) &= u_0(x)
\end{aligned}
$$

with the incompressibility condition

$$\nabla \cdot u = 0$$

- We impose periodic boundary conditions:

$$u(x+e_{x_i},t) = u(x,t),\ e_{x_i} \text{ unit vector in } \mathbb{R}^2 \text{ or } \mathbb{R}^3$$

- Eliminating the pressure using the incompressibility condition gives

$$
\begin{aligned}
u_t + u \cdot \nabla u &= \nu \Delta u + \nabla \Delta^{-1} \mathrm{trace}(\nabla u)^2 \qquad (1) \\
u(x,0) &= u_0(x)
\end{aligned}
$$

The Millennium Problem

- The Millennium Problem is: prove that there exists a smooth solution $u(x,t)$ of the initial value problem (1) in three-dimensions, $x \in \mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$
- In other words, show that there exists a divergence free function $u \in C^\infty(\mathbb{T}^3 \times [0,\infty))$ that satisfies the initial value problem, IVP (1)
- We need an infinitely differentiable function that both solves the Navier-Stokes equation and takes on the initial condition $u(x,0) = u_0(x)$, when $t = 0$

Weak Solutions

- In two dimensions, this problems has been solved, there exists a divergence free function $u \in C^\infty(\mathbb{T}^2 \times [0,\infty))$ that satisfies the IVP (1)
- The story is more complicated in three dimensions
- There exist weak solutions, but it is not known if they are unique [10] (1933), [5] (1951)

Jean Leray

Eberhard Hopf

Existence Proof

- Leray proved the a priori bounds

$$|u|_2^2(t) \leq |u_0|_2^2 e^{-2\lambda_1 \nu t}, \int_0^t |\nabla u|_2^2(s)ds \leq \frac{1}{2\nu}|u_0|_2^2,\ t \geq 0, \tag{2}$$

  where λ_1 is the first smallest eigenvalue of $-\Delta$ with vanishing boundary conditions on $\mathbb{T}^3$, and $|u|_2$ denotes the $L^2(\mathbb{T}^3)$ norm, $|u|_2^2 = \int_{\mathbb{T}^3} |u(x,t)|^2 dx$
- Then if we can show that (1) defines a contraction on the Sobolev space $H^1(\mathbb{T}^3)$ with norm $|\nabla u|_2$, then we are done
- This works in 2 dimensions, because $|u|_\infty \leq ||u||_k$ for $k > n/2$, but in three dimensions the argument fails

Uniqueness is Missing

- The IVP is well-posed, if there exists a unique solution that depends continuously on its initial data
- In dimension two, the Navier-Stokes IVP for viscous incompressible Newtonian fluid is well posed
- but not in dimension three
- Specifically, for the 3D Navier-Stokes equations the existence of global weak solutions is known since the works of Leray and Hopf
- but the uniqueness is an open problem
- the regular solutions are unique
- but the existence of global regular solutions is open

The a Priori Estimate in Two Dimensions

- We work with the vorticity $\omega = \nabla \times u$
- ω satisfies the same a priori estimates as u:

$$|\omega|_2^2(t) \leq |\omega_0|_2^2 e^{-2\lambda_1 \nu t}, \int_0^t |\nabla\omega|_2^2(s)ds \leq \frac{1}{2\nu}|\omega_0|_2^2, \; t \geq 0$$

- From the Navier-Stokes (vorticity) equation, and using $|u|_\infty \leq c_3|\Delta u|_2 \leq c_3|\nabla\omega|_2$, we can derive the equation

$$\frac{d|\nabla\omega|_2^2}{dt} \leq c_1|u|_\infty^2|\nabla\omega|_2^2 \leq c_2|\Delta u|_2^2|\nabla\omega|_2^2 \leq c|\nabla\omega|_2^4$$

- Now integrating in t and applying Grönwall's inequality we get

$$|\nabla\omega|_2^2(t) \leq Ce^{c\int_0^t |\nabla\omega|_2^2(s)ds} \leq Ce^{\frac{c}{2\nu}|\omega_0|_2^2} < \infty$$

Outline

Laminar and Turbulent Flow

The Reynolds Number : $Re = \frac{UL}{\nu}$

- In 1883 the mechanical engineer Osborne Reynolds observed:
  "The internal motion of water assumes one or other of two broadly distinguishable forms-either the elements of the fluid follow one another along lines of motion which lead in the most direct manner to their destination or they eddy about in sinuous paths the most indirect possible."
- These are respectively laminar and turbulent flow

The Reynolds Number

- O. Reynolds realized that which kind of flow one observed was a question of the ratio of the inertial and viscous forces
- He also identified the parameter, expressing this ratio, the Reynolds number,

$$Re = \frac{UL}{\nu} \tag{3}$$

  that determined whether the flow is laminar or turbulent
- Here U is a typical velocity of the flow, L is a typical length scale in the flow and ν is the kinematic viscosity
- Reynolds also found the onset of turbulence experimentally to be at $Re = 500$ and the fluid to be fully turbulent at $Re = 2000$

Are most flows in Nature Laminar or Turbulent?

Table: Reynolds Numbers of Different Flows

Type	Width (m)	Dept (m)	Velocity (m/s)	Re
Vertical Trickle	0.007	0.002	0.7	1087
Parking Lot	25	0.002	0.1	151.5
Model River	0.5	0.05	0.1	3.8×10^3
Small Stream	5	0.5	0.5	1.9×10^5
Medium River	25	2	1	1.5×10^5
Large River	100	5	1	3.8×10^6
Gulf Stream	4×10^4	2×10^3	1	1.5×10^9

Laminar $<<$ 500, Fully Turbulent $>$ 2000

The Kolmogorov-Obukhov Theory

- In 1941 Kolmogorov and Obukhov [8, 7, 11] proposed a statistical theory of turbulence
- The structure functions of the velocity differences of a turbulent fluid, should scale with the distance (lag variable) l between them, to the power $p/3$

$$E(|u(x,t)-u(x+l,t)|^p) = S_p = C_p l^{p/3}$$


A. Kolmogorov


A. Obukhov

The Kolmogorov-Obukhov Refinded Similarity with She-Leveque Intermittency Corrections

- The Kolmogorov-Obukhov '41 theory was criticized by Landau for including universal constants C_p and later for not including the influence of the intermittency
- In 1962 Kolmogorov and Obukhov [9, 12] proposed a refined similarity hypothesis

$$S_p = C'_p < \tilde{\varepsilon}^{p/3} > l^{p/3} = C_p l^{\zeta_p} \tag{4}$$

  l is the lag and ε a mean energy dissipation rate
- The scaling exponents

$$\zeta_p = \frac{p}{3} + \tau_p$$

  include the She-Leveque intermittency corrections $\tau_p = -\frac{2p}{9} + 2(1-(2/3)^{p/3})$ and the C_p are not universal but depend on the large flow structure

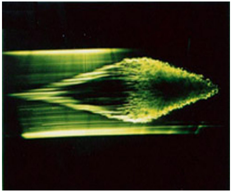
Onsager's Observation

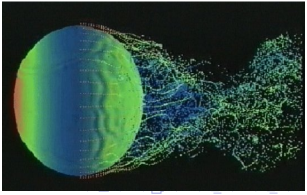
Lars Onsager

- In 1945 the physicist L. Onsager made a profound observation [13]
- Fluid velocity that satisfied the Kolmogorov-Obukhov scaling can not have a a gradient ∇u continuous in x
- The precise mathematical term is that u is Hölder continuous in x with Hölder index $1/3$
- This means that the turbulent solutions are not smooth
- The solution of the Millennium problem must be the laminar solution

The Difference between Laminar and Turbulent

- If the Reynolds number is small, only the laminar solution exists
- In this case the ambient noise is quelled
- If the Reynolds number is large, the laminar solution exists but is unstable
- The ambient noise is magnified by the instabilities of the laminar flow and becomes large
- Then the turbulent solution satisfies a stochastic partial differential equation (SPDE)





Outline

Stochastic Navier-Stokes with Turbulent Noise

- The stochastic Navier-Stokes equations describing fully developed turbulence [2, 3]

$$du = (\nu\Delta u - u\cdot\nabla u + \nabla\Delta^{-1}\mathrm{tr}(\nabla u)^2)dt$$

$$(\text{additive noise } =) + \sum_{k\neq 0}[c_k^{\frac{1}{2}} db_t^k e_k(x) + d_k|k|^{1/3}dt]\ e_k(x)$$

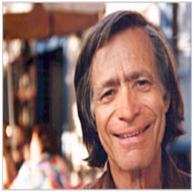
$$(\text{multiplicative noise } =) + u(\sum_{k\neq 0}^{M}\int_{\mathbb{R}} h_k\bar{N}^k(dt,dz))$$

$$u(x,0) = u_0(x)$$

- Each Fourier component $e_k = e^{2\pi k\cdot x}$ comes with its own Brownian motion b_t^k and deterministic bound $|k|^{1/3}dt$

Adding Noise

- Adding noise is not a new idea
- Kraichnan and Sinai both added noise to the Navier-Stokes equation to study turbulence
- The new idea was to use the Central Limit Theorem and the Large Deviation Principle to give the additive noise and pure jumps for the multiplicative noise


R. Kraichnan


Y. Sinai

Solution of the Stochastic Navier-Stokes

- We solve (5) using the Feynmann-Kac formula, and Cameron-Martin (or Girsanov's Theorem)
- The solution is

$$\begin{aligned} u &= e^{Kt} e^{\int_0^t dq} M_t u^0 \\ &+ \sum_{k\neq 0} \int_0^t e^{K(t-s)} e^{\int_s^t dq} M_{t-s} (c_k^{1/2} d\beta_s^k + d_k \mu_k ds) e_k(x) \end{aligned}$$

- K is the operator $K = \nu\Delta + \nabla\Delta^{-1} tr(\nabla u \nabla)$
- $M_t = exp\{-\int_0^t u(B_s, s) \cdot dB_s - \frac{1}{2}\int_0^t |u(B_s, s)|^2 ds\}$
- The Feynmann-Kac formula produces the log-Poisson processes of She and Leveque [14]

$$e^{\int_0^t dq} = e^{N_t^k \ln\beta + \gamma \ln|k|} = |k|^\gamma \beta^{N_t^k}$$

Computation of the structure functions

Lemma (The Kolmogorov-Obukhov-She-Leveque scaling)

The scaling of the structure functions is

$$S_p \sim C_p|x-y|^{\zeta_p},$$

where

$$\zeta_p = \frac{p}{3} + \tau_p = \frac{p}{9} + 2(1-(2/3)^{p/3})$$

$\frac{p}{3}$ *being the Kolmogorov scaling and* τ_p *the intermittency corrections. The scaling of the structure functions is consistent with Kolmogorov's* $4/5$ *law,*

$$S_3 = -\frac{4}{5}\varepsilon|x-y|$$

to leading order, were $\varepsilon = \frac{d\mathcal{E}}{dt}$ *is the energy dissipation*

KOSL Scaling of the Structure Functions, higher order $Re_\lambda \sim 16,000$

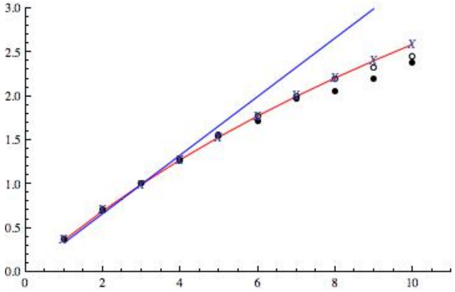
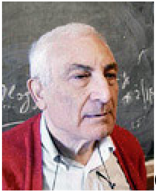


Figure: The exponents of the structure functions as a function of order, theory or Kolmogorov-Obukov-She-Leveque scaling (red), experiments (disks), dns simulations (circles), from [4], and experiments (X), from [14]. The Kolomogorov-Obukhov '41 scaling is also shown as a blue line for comparion.

Outline

The Statistical Theory of Turbulence


G. Da Prato


J. Zabczyk

The Invariant Measure and the Probability Density Functions (PDF)

- Hopf [6] write down a functional differential equation for the characteristic function of the invariant measure
- The Kolmogorov-Hopf equation for (5) is

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\mathrm{tr}[P_t C P_t^* \Delta \phi] + \mathrm{tr}[P_t \bar{D} \nabla \phi] + < K(z) P_t, \nabla \phi > \quad (5)$$

where $\bar{D} = (|k|^{1/3} D_k)$, $\phi(z)$ is a bounded function of z,

$$P_t = e^{-\int_0^t \nabla u \ dr} M_t \prod_k^m |k|^{2/3} (2/3)^{N_t^k}$$

- Variance and drift

$$Q_t = \int_0^t e^{K(s)} P_s C P_s^* e^{K^*(s)} ds, \quad E_t = \int_0^t e^{K(s)} P_s \bar{D} ds \quad (6)$$

The invariant measure of the stochastic Navier-Stokes

The solution of the Kolmogorov-Hopf equation (5) is

$$R_t\phi(z) = \int_H \phi(e^{Kt}P_t z + El + y)\mathcal{N}_{(0,Q_t)} * \mathbb{P}_{N_t}(dy)$$

Theorem

The invariant measure of the Navier-Stokes equation on $H_c = H^{3/2^+}(\mathbb{T}^3)$ is, $\mu(dx) =$

$$e^{<Q^{-1/2}El,\ Q^{-1/2}x>-\frac{1}{2}|Q^{-1/2}El|^2}\mathcal{N}_{(0,Q)}(dx)\sum_k \delta_{k,l}\sum_{j=0}^{\infty} p^j_{m_l}\delta_{(N_l-j)}$$

where $Q = Q_\infty$, $E = E_\infty$.

The Uniqueness of the Invariant Measure

- Finding the invariant measure solves the turbulence problem
- Namely, all the (deterministic) statistical properties of the turbulent velocity are determined by the invariant measure
- But is the invariant measure unique?
- No, but invariant measures corresponding to different weak solution are absolutely continuous with respect to each other
- Thus their statistical theories are equivalent
- The statistical theory is unique

The Probability Density Function (PDF)

Lemma

The PDF is a Normalized Inverse Gaussian distribution NIG of Barndorff-Nilsen [1]:

$$f(x_j) = \frac{(\delta/\gamma)}{\sqrt{2\pi}K_1(\delta\gamma)} \frac{K_1\left(\alpha\sqrt{\delta^2+(x_j-\mu)^2}\right) e^{\beta(x-\mu)}}{\left(\sqrt{\delta^2+(x_j-\mu)^2}/\alpha\right)} \tag{7}$$

where K_1 is modified Bessel's function of the second kind, $\gamma = \sqrt{\alpha^2-\beta^2}$.

$$f(x) \sim \frac{(\delta/\gamma)}{2\pi K_1(\delta\gamma)} \frac{\Gamma(1)2e^{\beta\mu}}{(\delta^2+(x-\mu)^2)}, \quad x << 1$$

$$f(x) \sim \frac{(\delta/\gamma)}{2\pi K_1(\delta\gamma)} \frac{e^{\beta(x-\mu)}e^{-\alpha x}}{x^{3/2}}, \quad x >> 1$$

O. Barndorff-Nilsen

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The log of the PDF from simulations and fits for the longitudinal direction

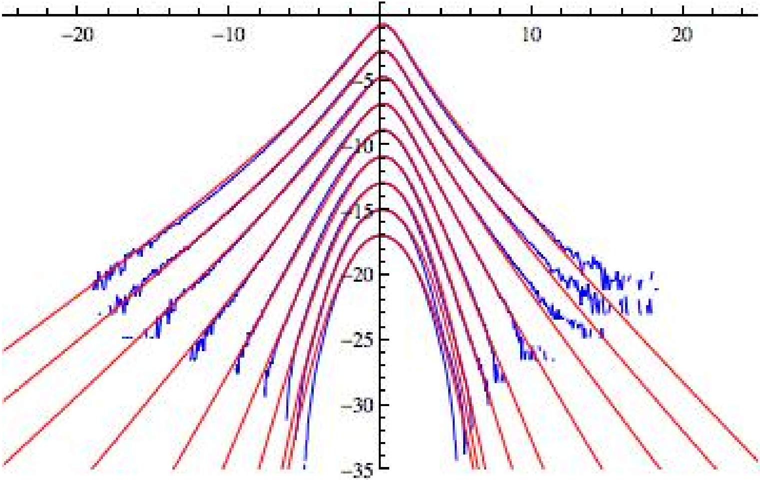

Figure: The log of the PDF from simulations and fits for the longitudinal direction, compare Fig. 4.5 in [15].

Outline

Back to the Millennium Problem

- What does the solution of the turbulence problem tell us about the laminar solution?
- It gives a whole arsenal of techniques to try to solve the millennium problem
- Probabilistic techniques can be used to solve linear deterministic equations such as the heat equation and Laplace's equation
- If we can use these methods to solve the millennium problem, then we can also use them to finds a unique solution (stochastic process) of the stochastic Navier-Stokes equation
- This will give us many more details about the solution to the turbulence problem

The Artist by the Water's Edge

Leonardo da Vinci Observing Turbulence

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