Math 119 A, Midterm

ODEs and Dynamical Systems

1. 10 pts. Write the ODE as a first order system

$$\frac{d^4x}{d^4t} + 5\frac{d^3x}{d^3t} + 7\frac{d^2x}{d^2t} + 2\frac{dx}{dt} + 4x = 0.$$

Solution:

$$x_1 = x,\ x_2 = \dot{x} = \dot{x}_1,\ x_3 = \dot{x}_2,\ x_4 = \dot{x}_3,$$

$$\dot{x}_4 = -5x_4 - 7x_3 - 2x_2 - 4x_1.$$

The first order system is

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & -2 & -7 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

2. Solve the 2x2 ODE systems $\dot{x} = Ax$ by finding the eigenvalues and eigenvectors and exponentiating the matrix $P^{-1}AP$,

(a) 5 pts.

$$A = \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix},$$

Solution: We first find the e-values:

$$det\begin{pmatrix} \lambda - 2 & -6 \\ -1 & \lambda + 3 \end{pmatrix} = \lambda^2 + \lambda - 12 = 0$$

The solutions are $\lambda = 3, -4$. Then we find the e-vectors:

$$v_1 = \begin{pmatrix} 6 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The transformation matrix is:

$$P = \begin{pmatrix} 6 & 1 \\ 1 & -1 \end{pmatrix}, \qquad P^{-1} = \frac{1}{7}\begin{pmatrix} 1 & 1 \\ 1 & -6 \end{pmatrix}.$$

The y space solution is:

$$y(t) = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-4t} \end{pmatrix} y_0$$

and the x space solution is:

$$x(t) = P\begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-4t} \end{pmatrix} P^{-1}x_0 = \frac{1}{7}\begin{pmatrix} 6e^{3t} + e^{-4t} & 6e^{3t} - 6e^{-4t} \\ e^{3t} - e^{-4t} & e^{3t} + 6e^{-4t} \end{pmatrix} x_0.$$

(b) 5 pts.

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix},$$

Solution: This matrix is already in Jordan normal form. Thus we can write the solution immediately,

$$x(t) = e^t\begin{pmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{pmatrix} x_0.$$

(c) 5 pts.

$$A = \begin{pmatrix} 1 & 5 \\ -\frac{1}{2} & 2 \end{pmatrix}.$$

Solution: We first find the e-values,

$$det\begin{pmatrix} \lambda - 1 & -5 \\ \frac{1}{2} & \lambda - 2 \end{pmatrix} = \lambda^2 - 3\lambda + \frac{9}{2} = 0$$

The solutions are complex conjugates $\lambda = \frac{3}{2} \pm \frac{3}{2}i$. Then we find the complex conjuage e-vectors

$$\begin{pmatrix} \frac{1}{2} + \frac{3}{2}i & -5 \\ \frac{1}{2} & -\frac{1}{2} + \frac{3}{2}i \end{pmatrix}\begin{pmatrix} a + ib \\ c + id \end{pmatrix} = 0,$$

$$b = -3a, \; c = \frac{a}{10} - \frac{3b}{10} = a, \; d = -\frac{3a}{10} - \frac{-b}{10} = 0.$$

Thus

$$v = \begin{pmatrix} 1 - 3i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} -3 \\ 0 \end{pmatrix} = u + iw.$$

The transformation matrix is:

$$P = [wu] = \begin{pmatrix} 1 & -3 \\ 1 & 0 \end{pmatrix}, \qquad P^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 \end{pmatrix}.$$

The Jordan normal for is:

$$P^{-1}AP = \begin{pmatrix} 1 & -3 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ -\frac{1}{2} & 2 \end{pmatrix}\begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -\frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} \end{pmatrix}.$$

In y space the solution is:

$$y(t) = e^{\frac{3}{2}t}x(t)\begin{pmatrix} \cos(\frac{3}{2}t) & -\sin(\frac{3}{2}t) \\ \sin(\frac{3}{2}t) & \cos(\frac{3}{2}t) \end{pmatrix}y_0,$$

and in x space it is:

$$x(t) = P^{-1}e^{\frac{3}{2}t}\begin{pmatrix} \cos(\frac{3}{2}t) & -\sin(\frac{3}{2}t) \\ \sin(\frac{3}{2}t) & \cos(\frac{3}{2}t) \end{pmatrix}Px_0$$

$$= e^{\frac{3}{2}t}\begin{pmatrix} \cos(\frac{3}{2}t) - \frac{1}{3}\sin(\frac{3}{2}t) & \frac{10}{3}\sin(\frac{3}{2}t) \\ -\frac{1}{3}\sin(\frac{3}{2}t) & \cos(\frac{3}{2}t) + \frac{1}{3}\sin(\frac{3}{2}t) \end{pmatrix}x_0.$$

3. Draw the phase portraits for the three ODEs above and classify the flow as sinks, sources, centers, etc.

   (a) 5 pts.

   $$A = \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix},$$

   A saddle.

   (b) 5 pts.

   $$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix},$$

   A spiral source.

   (c) 5 pts.

   $$A = \begin{pmatrix} 1 & 5 \\ -\frac{1}{2} & 2 \end{pmatrix}.$$

   A spiral source.

4. 10 pts. Suppose that a matrix A has one complex eigenvalue $\lambda = a+ib$, with $a < 0$ and one real eigenvalue $\lambda = c < 0$ with (algebraic) multiplicity two but only one eigenvector. What does the Jordan normal form $P^{-1}AP$ of the matrix look like and what is the solution $y(t)$ of the ODE

$$\dot{x} = Ax$$

in the coordinates $y = P^{-1}x$? What does the phase portrait look like? Solution: The Jordan normal form is:

$$\begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & c & 1 \\ 0 & 0 & 0 & c \end{pmatrix}.$$

The phase portrait is a spiral sink.

5. Solve the 3x3 ODE systems $\dot{x} = Ax$ by finding the eigenvalues and eigenvectors and exponentiating the matrix $P^{-1}AP$,

   (a) 10 pts.

$$A = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

Solution: We first find the e-values,

$$det \begin{pmatrix} \lambda - 2 & -4 & -1 \\ 0 & \lambda - 3 & -2 \\ 0 & 0 & \lambda - 1 \end{pmatrix} = (\lambda - 2)(\lambda - 3)(\lambda - 1) = 0,$$

$\lambda = 2, 4, 1$. The corresponding e-vectors are:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}.$$

The similarity (transformation) matrix is:

$$P = \begin{pmatrix} 1 & 4 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \qquad P^{-1} = \begin{pmatrix} 1 & -4 & -7 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

The Jordan normal form is:

$$P^{-1}AP = \begin{pmatrix} 1 & -4 & -7 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The solution in y space is:

$$y(t) = \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix} y_0,$$

and in x space,

$$x(t) = P \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix} P^{-1} x_0 = \begin{pmatrix} e^{2t} & 4e^{2t} - 4e^{3t} & -7e^{2t} + 4e^{3t} + 3e^{t} \\ 0 & e^{3t} & e^{3t} - e^{t} \\ 0 & 0 & e^{t} \end{pmatrix} x_0.$$

(b) 10 pts.

$$A = \begin{pmatrix} 2 & 1 & 0 \\ -9 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

Solution: This matrix falls into a 2 by 2 matrix and a scalar equation $\dot{x} = 2x$. Thus we can solve the 2 by 2 matrix and the scalar equation separately and then put the solutions together into a 3 by 3 matrix. The compute the e-values of the 2 by 2 matrix:

$$det \begin{pmatrix} \lambda - 2 & -1 \\ 9 & \lambda - 2 \end{pmatrix} = (\lambda - 2)^2 + 9 = 0$$

The solutions are complex conjugates $\lambda = 2 \pm 3i$. Next we compute the e-vectors:

$$\begin{pmatrix} 3i & -1 \\ 9 & 3i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives $b = 3ia$ or

$$v = \begin{pmatrix} 1 \\ 3i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 3 \end{pmatrix} = u + iw.$$

The similarity martix is:

$$P = [wu] = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}, \qquad P^{-1} = \begin{pmatrix} 0 & \frac{1}{3} \\ 1 & 0 \end{pmatrix}.$$

The Jordan normal form is:

$$P^{-1}\begin{pmatrix} 2 & 1 \\ -9 & 2 \end{pmatrix}P = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}.$$

Thus in y space we the the 3 by 3 equation

$$\dot{y} = \begin{pmatrix} 2 & -3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} y$$

with the solution

$$y(t) = e^{2t}\begin{pmatrix} \cos(3t) & -\sin(3t) & 0 \\ \sin(3t) & \cos(3t) & 0 \\ 0 & 0 & 1 \end{pmatrix} y_0$$

To get the solution in x space we transform the 2 by 2 matrix:

$$Pe^{2t}\begin{pmatrix} \cos(3t) & -\sin(3t) \\ \sin(3t) & \cos(3t) \end{pmatrix}P^{-1} = e^{2t}\begin{pmatrix} \cos(3t) & \frac{1}{3}\sin(3t) \\ -3\sin(3t) & \cos(3t) \end{pmatrix}$$

and get the 3 by 3 matrix

$$x(t) = e^{2t}\begin{pmatrix} \cos(3t) & \frac{1}{3}\sin(3t) & 0 \\ -3\sin(3t) & \cos(3t) & 0 \\ 0 & 0 & 1 \end{pmatrix} x_0.$$

(c) 10 pts.

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

Solution: Since this matrix is upper diagonal we can read the e-values of the diagonal. $\lambda = 2$ is an e-value with algebraic multiplicity 3. We find the e-vector:

$$\begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The single e-vector is:

$$v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

This means that the geometric multiplicity is one and we must find two generalized e-vectors. We first solve:

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = v$$

The solution is,

$$w_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

and v. Then we solve:

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = w_1,$$

to get

$$w_2 = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

and v. Now the similarity matrix is:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \qquad P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

The Jordan normal form is:

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

The solution in y space is:

$$y(t) = e^{2t} \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} y_0,$$

and in x space,

$$x(t) = Pe^{2t}\begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}P^{-1}x_0 = e^{2t}\begin{pmatrix} 1 & t & t+t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix}x_0.$$

6. 10 pts. Find the subspaces E^s, E^c, E^u of the ODE

$$\dot{x} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} x,$$

and find vectors x^s, x^c, x^u in each subspace. What happens to the solutions with these vectors as inital data as $t \to \infty$?
Solution:

$$\mathrm{E^u} = \mathrm{span}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \mathrm{E^c} = \mathrm{span}\{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\}, \ \mathrm{E^s} = \mathrm{span}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$\lim_{t\to\infty} x^u(t) = \infty, \ \lim_{t\to\infty} x^s(t) = 0,$$

$x^c(t)$ stays on a circle in phase space.

7. 10 pts. Find the Jordan normal form of the matrix and solve the IVP,

$$A = \begin{pmatrix} 0 & -1 & -2 & -1 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Solution: Problem 7 is Example 4 on page 45 in Perko and the solution is given there.